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ORIGINAL ARTICLE

Paraxial Propagation of Astigmatic Wavefronts in Optical Systems by an Augmented Stepalong Method for Vergences

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ABSTRACT: ***Purpose.*** **The propagation of astigmatic wavefronts through astigmatic optical systems is reconsidered in the wavefront perspective.** ***Methods.*** **The stepalong method for vergences, described by 2 × 2 matrices, is applied and augmented to produce off-axis information like the magnification. This so-called augmented stepalong method (ASAM) is derived by applying the paraxial propagation of astigmatic wavefronts to tilted wavefronts as well.** ***Results.*** **The features of the ASAM are discussed for a single surface, a thick lens, and a general system.** ***Conclusions.*** **The ASAM provides all necessary information to describe a centered astigmatic optical system in paraxial approximation. (Optom Vis Sci 2005;82:923–932)**

Key Words: astigmatic wavefronts, paraxial wavefront propagation, augmented stepalong method, vergence, magnification

Higher-order wavefront aberrations of the eye play a growing part in optometry. As a canonical language, so-called wavefront aberrations have emerged. For their representation, Zernike polynomials are the preferred dialect. Although the wavefront picture is inherent to basic optometric concepts, like vergence or refraction, the mainstream approach in optometry is based on light rays. The theory of linear optics of astigmatic systems, extensively developed by Harris,[1] Keating,[2] Long,[3] and Fick[4] (in reverse historical order), is based mainly on ray optics. The issue of wavefronts is rarely considered, with the exceptions of Harris[5] and Thibos.[6]

In the perspective of geometric optics, both concepts, namely ray and wave optics, are 2 sides of one and the same coin. Wavefronts are normal to rays and rays are orthogonal to wavefronts. It appears just as a question of use which tool is applied to describe and solve a problem. Light rays have won this competition in many cases and are probably one of the most familiar concepts in optics. They are well known even from middle school where the concept of graphic ray tracing makes geometric optics comprehensible. When, later on, it comes to optometry, light rays are still the main tool. It seems that a hybrid situation in ophthalmic optics has occurred: We are applying wavefront-based terminology, like vergence or refraction, in everyday work. To understand what we are doing, we switch to linear *ray* optics of astigmatic systems. Eventually, if it comes to higher-order aberrations, we reswitch to the wavefront language again. The first purpose of this article is to bridge this gap by reconsidering the paraxial propagation of astigmatic wavefronts. Harris did something very similar.[5] He started from the ray optics concept and derived by integration of differential relations the very concept of an astigmatic wavefronts and their properties. Our starting point is the description of electromagnetic scalar waves. We apply what is known as Fresnel propagation to rederive well-known results. Some, as we believe interesting, aspects show up along this way.

The second purpose is to provide a method for including off-axis information into the well known stepalong procedure for vergences. In the simplest case of paraxial optics of rotationally symmetric systems, which is called Gaussian optics in the following section, the knowledge of 2 quantities determines the properties of the imagery: position and size of the image.

When it comes to a general optical system, like the optics of the eye, we have to switch from Gaussian optics to the linear optics of an astigmatic system, which still works in the paraxial domain. Excluding decentered or tilted systems, everything that can happen to a ray is described by a 4 × 4 system matrix, also called the transference. This matrix is generated by the repeated multiplication of 4 × 4 matrices related to the 2 events in the life of a light ray: transfer (free propagation) and refraction. If the transference is known the linear mapping, which connects all possible ray heights and ray angles in 2 reference planes is completely characterized.

Therefore, all information about the optical system between the reference planes is contained in this 4×4 matrix. Clearly, there are 16 numbers in a 4×4 matrix. However, not all of them are independent. The transference matrix can not be filled with arbitrary numbers, because certain relations, known as symplectic relations, have to be obeyed. Therefore, a maximum of only 10 of the 16 components are independent. Then obviously the transference contains redundant information. However, still 10 numbers are much more than the 2 quantities known from simple Gaussian optics. Harris[7] partitioned the transference into 2×2 submatrices called dilation, disjugacy, divergence, and divarication to name, illustrate, and discuss the effects of special transference elements on the imagery. All this is based on the ray optics paradigm.

Instead of the transference often the so called stepalong method (SAM), which deals with 2×2 matrices only, is applied. This method exploits the wavefront picture, and it is used to calculate the wavefront curvature at successive points through an optical system. Compared with the handling of 4×4 matrices, it is a neat method to evaluate relevant on-axis information of a given centered optical system. Keating[8] gives many examples how to make use of this method. The disadvantage of the SAM, however, is the hitherto missing off-axis information. Nothing can be said on magnification for example. This situation is quite different from Gaussian optics in which the on-axis ray tracing produces the lateral magnification simultaneously.

As long as monocular vision is considered, magnification is not a big issue, because the key goal is to see "sharply." Therefore, from the known refraction of an eye, the on-axis wavefront is corrected. No recurrence to magnification is necessary for that purpose. If, however, the right and the left eyes' images differ in perceived size or shape we speak of (static) aniseikonia. Patients at risk for aniseikonia may belong to the group having anisometropia. An increasing number at risk are patients who had a cataract or refractive surgery. Pseudophakes, for example, may have complaints related to aniseikonia. Therefore, the magnification of an optical system and clearly the difference between those of 2 systems is an important quantity. A general and thorough approach by Harris[7] is available, which again is based on ray optics.

Based on the wavefront approach, we will show how an augmented stepalong method (ASAM) will support the missing link to off-axis information like magnification. This will be done by evaluating information available from the on-axis stepalong method only and with reference to the picture of propagating wavefronts. There are some advantages of the ASAM. First, the amount of numerical calculations is reduced, because only 2×2 matrices are involved. Furthermore, we believe that this method might be more comprehensive and compact, because there are less optical quantities involved than in the ray optics approach.

The article is organized as follows. The next part includes a short review of the basics of paraxial wavefront propagation. We start with the well-known refraction at a single interface and then give a fresh look at the free propagation of a wavefront, which usually is called the transfer. To this end, we consider the diffraction approach in the Fresnel approximation. To our knowledge, this has not been done before in this way. As seen later, the odd cases of magnification with nonorthogonal axes of magnification cannot be produced by refraction at interfaces but are the result of the transfer process.

Then, in the kernel section, we augment the stepalong method with off-axis information by considering shifted or tilted wavefronts at one interface. In the remaining parts, we consider more complex systems. We start with the thick lens and continue with the discussion of general systems. The results are discussed in the last section. An illustrating numeric example for a simple pseudophakic eye model is found in the appendix.

Paraxial Wavefront Propagation

Waves are described by a sinusoidal or complex exponential function. The argument of this function is called the phase of the wave. All points connecting positions, in which the phase has a constant value, are called a wavefront. A wavefront may be described in an explicit way in a convenient coordinate system, say

$$z = W(x, y) = W(\mathbf{r}) \tag{1}$$

where z is the sagitta of the wavefront as a function of coordinates $\mathbf{r} = (x,y)^T$. The paraxial approximation will be introduced by a Taylor-expansion up to and including second-order terms

$$\begin{aligned} W(\mathbf{r}) = W_0 + x\left(\frac{\partial W(\mathbf{r})}{\partial x}\right)_{\mathbf{r}=\mathbf{r}_0} + y\left(\frac{\partial W(\mathbf{r})}{\partial y}\right)_{\mathbf{r}=\mathbf{r}_0} + \\ x^2\left(\frac{\partial^2 W(\mathbf{r})}{\partial x^2}\right)_{\mathbf{r}=\mathbf{r}_0} + 2xy\left(\frac{\partial^2 W(\mathbf{r})}{\partial x\,\partial y}\right)_{\mathbf{r}=\mathbf{r}_0} + \\ y^2\left(\frac{\partial^2 W(\mathbf{r})}{\partial y^2}\right)_{\mathbf{r}=\mathbf{r}_0} + \ldots \end{aligned} \tag{2A}$$

or equivalently

$$W(\mathbf{r}) = W_0 + \mathbf{r}^T \cdot (\nabla W)_0 + \frac{1}{2}\mathbf{r}[(\nabla\nabla^T)W]_0\mathbf{r} + \ldots . \tag{2B}$$

where

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)^T,$$

and $(\nabla\nabla^T)$ is the dyadic product leading to a matrix of second-order derivatives. All derivatives have to be evaluated at $\mathbf{r}_0 = 0$. We are dealing with 2-dimensional vectors and 2×2 matrices only and apply small letters to vectors and capital letters to matrices. A transposed quantity is denoted by a superscript T.

The piston or overall constant term W_0 has no significance concerning optical properties and will frequently be omitted in the following. The coefficient of the linear term, $(\nabla W)_0$ describes the direction or tilt of the wavefront at $\mathbf{r} = 0$. The expression $(\nabla\nabla^T)$W is called the Hessian matrix and is symmetric by definition. It is related to the principal curvatures and the axis of the principal meridians of the wavefront. To remember the paraxial nature, we will use a small letter w if the second-order approximation of the wavefront is used.

As an example, we consider a spherical wavefront with radius R. Related to the z-axis, we apply the well-known formula of a sphere to describe the wavefront by

$$W(x, y) = R\left(1 - \sqrt{1 - \frac{x^2 + y^2}{R^2}}\right) \quad (3)$$

After a Taylor expansion up to and including second-order terms and introducing the curvature c=1/R , we arrive at

$$w(x, y) = \frac{c}{2}(x^2 + y^2) \quad (4)$$

which is the well-known sagitta formula for given coordinates x and y. In anticipation of more general cases, we rewrite this expression as a quadric form in reduced quantities

$$\omega = nw = \frac{1}{2}(x \quad y)\begin{pmatrix} nc & 0 \\ 0 & nc \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\mathbf{r}^T\mathbf{L}\mathbf{r} \quad (5)$$

where the local refractive index n and the matrix of the reduced wavefront curvature $\mathbf{L}$ have been introduced.

Any sufficient smooth wavefront can locally be described by 3 numbers, which feature the intrinsic properties of a surface in a small patch around the chosen origin: 2 principal curvatures and a direction for one of the principal sections.[9] These 3 numbers could be translated into sphere, cylinder, and axis. Alternatively, the Hessian matrix of the second derivatives can be applied to describe an astigmatic wavefront by the more general reduced wavefront curvature

$$\mathbf{L} = \begin{pmatrix} S + C\sin^2\alpha & -C\cos\alpha\sin\alpha \\ -C\cos\alpha\sin\alpha & S + C\cos^2\alpha \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} \\ L_{12} & L_{22} \end{pmatrix} \quad (6)$$

where $S = nc_1$, $C = n(c_2 - c_1)$, represent a combination of the principal curvatures and α the orientation of c_1.

To follow up the wavefront transformations through an optical system, we need the description of 2 elementary operations: refraction and propagation of a wavefront. We recall the case of refraction, which has been dealt with in previous publications.[10–12] A refractive interface with principal curvatures κ_1, κ_2, and angle β of the principal meridian separating 2 media with refractive indices n and n' acts on an incoming wavefront as a phase transformer.[11] In the paraxial case in which the normal onto the wavefront and onto the surface are close together, the wavefront curvature matrix is transformed according to the well-known equation

$$\mathbf{L}' = \mathbf{F} + \mathbf{L} \quad (7)$$

where the dioptric power matrix of the surface[10] is given by

$$\mathbf{T} = (n' - n)\begin{pmatrix} \kappa_1 + (\kappa_2 - \kappa_1)\sin^2\beta & -(\kappa_2 - \kappa_1)\cos\beta\sin\beta \\ -(\kappa_2 - \kappa_1)\cos\beta\sin\beta & \kappa_1 + (\kappa_2 - \kappa_1)\cos^2\beta \end{pmatrix} \quad (8)$$

The picture is completed by the second step, which deals with the free propagation of a wavefront through a medium of refractive index n.

In the following, the propagation of an astigmatic wavefront is rederived by the diffraction approach in the Fresnel-approximation.[11] Assume an astigmatic wavefront described at a given plane, $z = 0$, by

$$\omega_0(\mathbf{r}_0, z = 0) = \frac{1}{2}[L_{11}x_0^2 + 2L_{12}x_0y_0 + L_{22}y_0^2] = \frac{1}{2}\mathbf{r}_0^T\mathbf{L}\mathbf{r}_0 \quad (9)$$

We apply the Fresnel propagation kernel to calculate the wavefront $\omega(\mathbf{r},z)$ at an axial position $z = t$ yielding

$$\exp(ik\omega(\mathbf{r}, z = t)) \propto \int \exp(ik\omega(\mathbf{r}_0))\exp\left(-\frac{ik}{2t}[\mathbf{r} - \mathbf{r}_0]^2\right)dx_0dy_0 \quad (10)$$

where $k = \frac{2\pi}{\lambda}$. This kind of integral can be solved by an asymptotic expansion.[13] To this end, the so-called critical points of 1st kind, x_c, y_c, have to be determined by the conditions

$$\frac{\partial}{\partial x_c}\left(\frac{1}{2}L_{11}x_c^2 + L_{12}x_cy_c + \frac{1}{2}L_{22}y_c^2 - \frac{1}{2t}[(x - x_c)^2 + (y - y_c)^2]\right) = 0$$
$$\frac{\partial}{\partial y_c}\left(\frac{1}{2}L_{11}x_c^2 + L_{12}x_cy_c + \frac{1}{2}L_{22}y_c^2 - \frac{1}{2t}[(x - x_c)^2 + (y - y_c)^2]\right) = 0 \quad (11)$$

or equivalently

$$L_{11}x_c + L_{12}y_c + \frac{1}{t}[x - x_c] = 0$$
$$L_{12}x_c + L_{22}y_c + \frac{1}{t}[y - y_c] = 0 \quad (12)$$

These equations may be interpreted as ray equations, and the linear ray optics approach has its starting point here. These linear relations can be rewritten as a matrix equation

$$\mathbf{T}\,\mathbf{r}_c = \mathbf{r} \quad (13)$$

where we introduced the matrix

$$\mathbf{T} = \mathbf{I} - t\mathbf{L} \quad (14)$$

and the 2 × 2 diagonal identity matrix $\mathbf{I}$. If $\mathbf{T}$ is regular, we can solve for the critical values as a function of $\mathbf{r}$:

$$\mathbf{r}_c = \mathbf{T}^{-1}\mathbf{r} \quad (15)$$

where

$$\mathbf{T}^{-1} = \frac{1}{\Delta}(\mathbf{I} - t(\det \mathbf{L})\cdot\mathbf{L}^{-1}) \quad (16)$$

being

$$\Delta = 1 - t\,\mathrm{tr}\mathbf{L} + t^2(\det\mathbf{L}) \quad (17)$$

The determinant and the trace of the matrix $\mathbf{L}$ are given by

$$\det\mathbf{L} = L_{11}L_{22} - L_{12}^2 \quad (18)$$

$$\mathrm{tr}\mathbf{L} = L_{11} + L_{22} \quad (19)$$

It is worth noting that $\mathrm{tr}\mathbf{L} = 2S + C$ and $\det\mathbf{L} = S(S + C)$ are both rotationally invariant quantities and therefore Δ has the same property. In case of a singular matrix $\mathbf{T}$, we have $\Delta = 0$ and the wavefront is propagating to one of the focal lines. Once the critical points have been evaluated to calculate the wavefront at the axial position , the following expression has to be evaluated at the critical points[13]

$$\omega(\mathbf{r};z = t) = \frac{1}{2}\mathbf{r}_c^T\mathbf{L}\mathbf{r}_c - \frac{1}{2t}(\mathbf{r} - \mathbf{r}_c)^2 \tag{20}$$

Applying equation 16 and some algebra leads to

$$\omega(\mathbf{r};z = t) = \frac{1}{2}\mathbf{r}^T\mathbf{L}^*\mathbf{r} \tag{21}$$

where

$$\mathbf{L}^* = \frac{1}{\Delta}\mathbf{L} - \frac{t}{\Delta}(\det \mathbf{L})\mathbf{I} \tag{22}$$

As we can see, the matrix $\mathbf{L}^*$, describing the vergence of the propagated wavefront at the end point, can be calculated as a linear combination of the vergence at the starting point $\mathbf{L}$ and the unity matrix $\mathbf{I}$. This result offers an alternative to the usual transfer equations discussed by Harris.[14] It is easy to show that our final equation may be also cast in a more familiar shape

$$\mathbf{L}^* = \frac{\mathbf{L}}{\mathbf{I} - t\mathbf{L}} \tag{23}$$

The equivalence of both may be proven by an elementwise calculation. Because $\mathbf{L}(\mathbf{I}-t\mathbf{L})^{-1} = (\mathbf{I}-t\mathbf{L})^{-1}\mathbf{L}$, the order of both matrices is irrelevant and may be abbreviated by the introduced fractional description, which is applied frequently in the forthcoming sections. The question of singular cases is shifted to the appendix, where detailed calculations for cases where a focal line coincides with an interface, are given.

Finally, and within this section, it is worth to point out the following mathematical result related with the Fresnel approach for propagation of wavefronts through an integral formulation.

If the astigmatic wavefront described in equation 9 were shifted parallel to the z-axis, then it could be described as

$$\begin{aligned}\omega_0(\mathbf{r}_0,\mathrm{z} = 0) &= \frac{1}{2}\,[\mathrm{L}_{11}(\mathrm{x}_0-\mathrm{x}_s)^2 + 2\mathrm{L}_{12}(\mathrm{x}_0-\mathrm{x}_s)(\mathrm{y}_0-\mathrm{y}_s) + \mathrm{L}_{22}(\mathrm{y}_0-\mathrm{y}_s)^2] \\ &= \frac{1}{2}(\mathbf{r}_0-\mathbf{r}_s)^{\mathrm{T}}\mathbf{L}(\mathbf{r}_0-\mathbf{r}_s)\end{aligned} \tag{24}$$

$\mathbf{r_s}$ being the amount of displacement or shift in the x **and** y coordinates. Then the propagation to $z = t$ can be easily evaluated by performing a change of variables in equation 10 and proceeding in the same way, yielding

$$\omega_0(\mathbf{r}_0, z = t) = \frac{1}{2}(\mathbf{r}_0 - \mathbf{r}_s)^T\mathbf{L}^*(\mathbf{r}_0 - \mathbf{r}_s) \tag{25}$$

with T^* as described in equation 22 or 23.

Augmented Stepalong Method

Off-axis information will be generated if we consider a *tilted* wavefront, which is no longer traveling along the z-axis. In this section, we see that in paraxial approximation, an off-axis point is equivalent to a shifted wavefront the axis of which is parallel to the z-axis. Because this fact is the central point of all forthcoming derivations, let us discuss first a simple case to illustrate this approach. Consider an off-axis point source placed at (x_s,R) with $R < 0$ as shown in Figure 1A. The wavefront passing at $x = 0, z = 0$ can be described by the equation of the sphere

$$(x - x_s)^2 + y^2 + (z - R)^2 = R^2 + x_s^2 \tag{26}$$

Solving for z and expanding the square root leads us to the equation of the tilted wavefront

$$w_T = z = \frac{1}{2}c(x^2 + y^2) - cx_s x \tag{27}$$

where constant terms have been omitted. Now, as shown in Figure 1B, consider a paraxial wavefront shifted by x_s, which reads

$$w_S = \frac{1}{2}[(x - x_s)^2 + y^2] = \frac{1}{2}c(x^2 + y^2) - cx_s x \tag{28}$$

Both equations, 27 and 28, again omitting constant terms, are identical, $w_T = w_S$. In other words, restricted to the paraxial approximation, the operations of shift and tilt lead to the same result (see Figure 1C).

The general case is represented by the following shifted wavefront

$$\omega_{shift} = \frac{1}{2}(\mathbf{r} - \mathbf{r}_s)^T\mathbf{L}(\mathbf{r}_0 - \mathbf{r}_s) \tag{29}$$

We expand this expression yielding

$$\omega_{shift} = \frac{1}{2}(\mathbf{r}^T\mathbf{L}\mathbf{r} - 2\mathbf{r}^T\mathbf{L}\mathbf{r}_s + \mathbf{r}_s^T\mathbf{L}\mathbf{r}_s) \tag{30}$$

where we applied the relation $\mathbf{r}^T\mathbf{L}\mathbf{r}_s = \mathbf{r}_s^T\mathbf{L}\mathbf{r}$, which holds for the symmetric vergence matrix $\mathbf{L}^T = \mathbf{L}$. The last constant term will again be omitted. Then we can rewrite the result as

$$\omega_{shift} = \omega - \mathbf{r}^T\mathbf{p} \tag{31}$$

where the vector $\mathbf{p}$ of optical direction angles (angles multiplied by the refractive index)

$$\mathbf{p} = \mathbf{L}\mathbf{r}_s \tag{32}$$

has been introduced. In other words, the off-axis wavefront ω_{shift} can be described by 2 components: the on-axis wavefront $\omega$and a tilt leading to the direction $\mathbf{p}/n$ at the z-axis. In case of a spherical wavefront, we are considering an off-axis point whose distance from the axis is given by the coordinates x_s and y_s and whose distance from the coordinate center is R. In case of a distant object point, the radius R goes to infinity and the components of r_s as well. The ratios of both, however, remain finite and result in the direction angle.

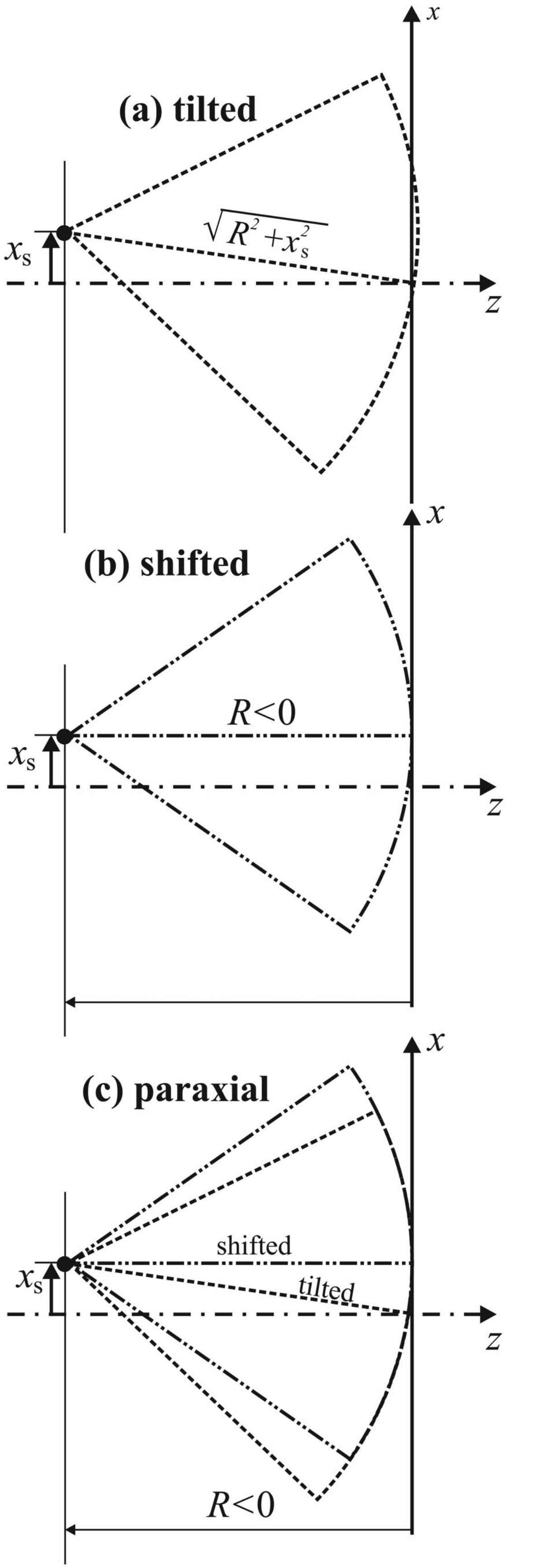

FIGURE 1.
Comparison of a tilted (dashed line, A) and a shifted (double dot dashed, B) wavefront in the *x*-*z*- plane. In the paraxial approximation (C), both modifications are equivalent.

Single Interface

In the following, we reconsider the stepalong method for vergences and include shifted wavefronts. First, we deal with the case of refraction at an interface characterized by the dioptric power matrix **F**. The connected phase change ω_F, introduced by the interface, has to be added to the phase ω of the incoming wavefront yielding the refracted wavefront

$$\omega' = \omega + \omega_F \tag{33}$$

or

$$\omega' = \frac{1}{2}\mathbf{r}^T\mathbf{L}\mathbf{r} - \mathbf{r}^T\mathbf{L}\mathbf{r}_s + \frac{1}{2}\mathbf{r}^T\mathbf{F}\mathbf{r} \tag{34}$$

Care has to be taken to interpret this expression. To evaluate the shift $\mathbf{r}_S'$ of the refracted wavefront, we rewrite the result as

$$\omega' = \frac{1}{2}\mathbf{r}^T\mathbf{L}'\mathbf{r} - \mathbf{r}^T\mathbf{L}'\mathbf{r}'_s \tag{35}$$

The shift in image space is then given by the relation

$$\mathbf{r}'_s = [(\mathbf{T}')^{-1}\mathbf{L}]\mathbf{r}_s = \mathbf{M}\mathbf{r}_s \tag{36}$$

where the existence of the inverse matrix $(\hat{\mathbf{L}}')^{-1} = (\mathbf{L} + \mathbf{F})^{-1}$ is assumed. The introduced *lateral* magnification **M** relates the shifts in image and object space and is given by

$$\mathbf{M} = (\mathbf{L}')^{-1}\mathbf{L} = \frac{\mathbf{L}}{\mathbf{L}'} \tag{37}$$

Again, the order of both matrices is irrelevant: $(\mathbf{L}')^{-1}\mathbf{L} = \mathbf{L}(\mathbf{L}')^{-1}$. The symbolic fraction reduces directly to the familiar scalar magnification in the case of spherical wavefronts. It is worth mentioning that these results do not require an imagery condition. In other words, the definition of the magnification **M** does not depend on the fact, that $\mathbf{r}_s$ and $\mathbf{r}_S'$ are conjugated to each other.

In addition to the lateral magnification, we now introduce an *angular* magnification **N**, which connects optical direction angles in object and image space. Note that the geometric angles have to be multiplied by the refractive indices. According to equations 32, the optical direction angles in object and image space are given by

$$\mathbf{p} = \mathbf{L}\mathbf{r}_s \quad \mathbf{p}' = \mathbf{L}'\mathbf{r}'_s \tag{38}$$

where $\mathbf{r}'_s$ is defined in equation 36. Both vectors of optical angles are now related by

$$\mathbf{p}' = \mathbf{N}\mathbf{p} \tag{39}$$

where the matrix of angular magnification **N** has to be determined. To this end, we insert equations 36 and 38 yielding

$$\mathbf{p}' = \mathbf{L}'\mathbf{r}'_s = \mathbf{L}'\mathbf{M}\mathbf{r}_s = \mathbf{L}'\mathbf{M}\mathbf{L}^{-1}\mathbf{p} \tag{40}$$

Therefore, we have the following relation between the angular and lateral magnification

$$\mathbf{N} = \mathbf{L}'\,\mathbf{M}\mathbf{L}^{-1} \tag{41}$$

or equivalently

$$\mathbf{M} = (\mathbf{L}')^{-1}\mathbf{N}\mathbf{L} \tag{42}$$

It is worth noting that the angular magnification is well defined for near objects and images as well and is not restricted to cases in which object and image are at infinity. However, in these cases, the lateral magnification cannot be applied and the angular magnification is of special use.

Given a single interface (a surface or a thin lens), the angular magnification simply reduces to $\mathbf{N} = \mathbf{I}$, as can be seen from equation 41. Then both optical direction angles are the same, $\mathbf{p}' = \mathbf{p}$. Because the related geometric angles are measured related to the *z*-axis, which is the surface normal at the same time, we have the paraxial version of the law of refraction (see Fig. 2).

The next step to be taken here is the propagation of a shifted wavefront, which has been already solved in equations 24 and 25. The wavefront

$$\omega = \frac{1}{2}\mathbf{r}^T\mathbf{L}[\mathbf{r} - 2\mathbf{r}_s] \tag{43}$$

while traveling a reduced distance *t* is described by

$$\omega(z = t) = \frac{1}{2}\mathbf{r}^T\mathbf{L}^*[\mathbf{r} - 2\mathbf{r}_s] \tag{44}$$

In other words, the shift is not modified by a propagation.

Summarizing the results of this section, we can state the obvious: the direction of a wavefront is not changed by free propagation, but by refraction. The basic quantity describing this change is the angular magnification, which is a matrix that can be calculated from on-axis vergences only.

Decentering of a Surface and Prentice's Equation

Because the following reverse case connects nicely to optometric terminology, we consider a shift of the lens by an amount $\mathbf{r}_L$. In other words, we decenter the lens. The phase change introduced by the lens, which has to be added to the incoming wavefront, is then given by

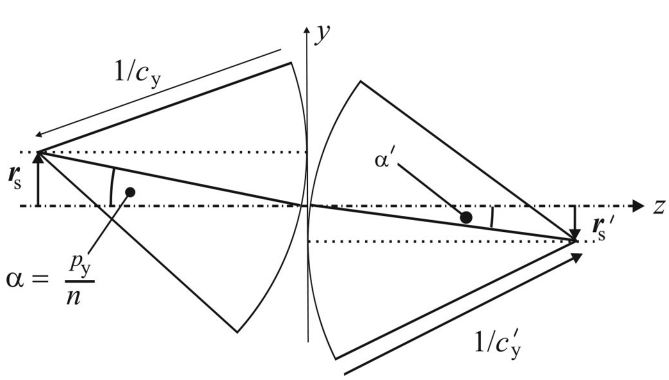

FIGURE 2.

Refraction of a wavefront in the *y-z-* plane. The curvatures of the wavefront are denoted by $c_y < 0$ and $c'_y > 0$. By multiplication with the refractive indices *n* and *n'*, we get the vergences in object and image space. The shift of the wavefront $\mathbf{r}_s$ is equivalent to a tilt angle $\alpha = p_y / n$ (<0), in which *n* is the refractive index and p_y the (optical) angle measured in the *y-z* plane relative to the *z*-axis.

$$\omega_F = \frac{1}{2}(\mathbf{r} - \mathbf{r}_L)^T\mathbf{F}(\mathbf{r} - \mathbf{r}_L) \tag{45}$$

leading to the outgoing wavefront

$$\omega' = \frac{1}{2}\mathbf{r}^T(\mathbf{L} + \mathbf{F})[\mathbf{r} - 2(\mathbf{L} + \mathbf{F})^{-1}\mathbf{F}\mathbf{r}_L] \tag{46}$$

with a shift

$$\mathbf{r}'_s = (\mathbf{L} + \mathbf{F})^{-1}\mathbf{F}\mathbf{r}_L \tag{47}$$

According to this shift, the direction of the outgoing wavefront is given by

$$\mathbf{p}' = (\mathbf{L} + \mathbf{F})\mathbf{r}'_s = \mathbf{F}\mathbf{r}_L \tag{48}$$

arriving at the well-known Prentice's equation.

It might be worth mentioning that physically the shift or decentering of the lens has no impact on the shape of the refracted wavefront. However, because the reference frame is shifted as well, we get a wavefront tilt related to $\mathbf{p}'$. From the wavefront's perspective, Prentice's rule is not related to the deviation of a single ray, but to the deviation of the entire collection of rays making up this wavefront.

Thick Lens

The ASAM is now extended from a single interface to the case of a thick lens or 2 separated thin lenses. The 2 interfaces have dioptric power matrices $\mathbf{F}_1$ and $\mathbf{F}_2$ and are separated by a medium with refractive index *n* and a reduced distance *t*. The stepalong method leads us to the following recursive relations

$$\mathbf{L}'_1 = \mathbf{L}_1 + \mathbf{F}_1 \tag{49}$$

$$\mathbf{L}_2 = \frac{\mathbf{L}'_1}{\mathbf{I} - t\mathbf{L}'_1} \tag{50}$$

$$\mathbf{L}'_2 = \mathbf{L}_2 + \mathbf{F}_2 \tag{51}$$

which support us with the traditional on-axis information. The lateral magnification is given by

$$\mathbf{M} = \frac{\mathbf{L}_2}{\mathbf{L}'_2}\frac{\mathbf{L}_1}{\mathbf{L}'_1} \tag{52}$$

After applying equations 49 to 51, we are led to the result

$$\mathbf{M} = [\mathbf{L}_1(\mathbf{I} - t\mathbf{F}_2) + \mathbf{F}^T]^{-1}\mathbf{L}_1 \tag{53}$$

where $\mathbf{F}^T = \mathbf{F}_1 + \mathbf{F}_2 - t\,\mathbf{F}_1\mathbf{F}_2$. Note that the magnification $\mathbf{M}$ is not a symmetric matrix in the general case. Again, the equation for the magnification reduces to the correct scalar equation.

For a distant object, we apply the relation $\mathbf{L}_1\mathbf{r}_s = \mathbf{p}$ while $\mathbf{L}_1 \rightarrow \mathbf{0}$ and arrive at

$$\mathbf{r}'_s = \mathbf{M}_\infty\mathbf{p} \tag{54}$$

The matrix $\mathbf{M}_\infty$ for a distant object, indicated by the right subscript ∞, is no longer dimensionless, because angles are transformed into positions. Equation 53 reduces now to

$$\mathbf{M}_\infty = (\mathbf{F}^T)^{-1} \tag{55}$$

and $\mathbf{M}_\infty$ is given by the inverse of the transposed dioptric power of the thick lens. In general, this is not a symmetric matrix and the principal axes might not be orthogonal. This result is well known from the transference approach, in which the imagery condition and the symplectic relations are exploited to arrive at this result.

In the case of an infinite image distance, we arrive by a similar reasoning at the following result

$$\mathbf{p}' = {}_\infty\mathbf{M}\mathbf{r}_s \tag{56}$$

where

$${}_\infty\mathbf{M} = \frac{\mathbf{L}_1}{\mathbf{I} - t(\mathbf{F}_1 + \mathbf{L}_1)} \tag{57}$$

The angular magnification, which is not only of use if both object and image are distant, is determined by

$$\mathbf{N} = \frac{\mathbf{L}_2}{\mathbf{L}'_1} = \frac{1}{\mathbf{I} - t\mathbf{L}'_1} = \frac{1}{\mathbf{I} - t(\mathbf{F}_1 + \mathbf{L}_1)} \tag{58}$$

which reduces to the well-known shape factor for $\mathbf{L}_1 = \mathbf{0}$. As seen below, the angular magnification has a more fundamental meaning than the lateral magnification.

General System

We now take the step to a general centered system with $k = 1,\ldots,K$ surfaces and K-1 intermediate spaces with reduced thicknesses t_k. At a surface with index k, the following relation holds for the vergence matrices before and after refraction

$$\mathbf{L}'_k = \mathbf{L}_k + \mathbf{F}_k \tag{59}$$

In front of the next surface labeled by $k+1$, we need to know the incoming vergence, which is given by

$$\mathbf{L}_{k+1} = \frac{\mathbf{L}'_k}{\mathbf{I} - t_k\mathbf{L}'_k} \tag{60}$$

The refraction at this surface leads to the vergence

$$\mathbf{L}'_{k+1} = \frac{\mathbf{T}'_k}{\mathbf{I} - t_k\mathbf{L}'_k}\mathbf{T}\mathbf{F}_k \tag{61}$$

This equation is rewritten as

$$\mathbf{L}'_{k+1}(\mathbf{I} - t_k\mathbf{L}'\,\mathbf{T}'_k) = \mathbf{L}'_k + \mathbf{F}_k(\mathbf{I} - t_k\mathbf{L}'_k) \tag{62}$$

Summing up all equations for $k = 1,\ldots,K$-1 leads after some arithmetic to the following result

$$\mathbf{L}'_K - \mathbf{L}_1 = \sum_{k=1}^{K}\mathbf{F}_k + \sum_{k=1}^{K-1} t_k\mathbf{L}_{k+1}\mathbf{L}'_k \tag{63}$$

which describes the total change in the wavefront given by the difference of the outgoing and incoming vergence matrices. The right side of the equation is made up of 2 sums. The first comprises the dioptric power matrices of all interfaces, whereas the second sum is related to the intermediate spaces. If all reduced thicknesses t_k are zero, we have an ensemble of juxtaposed interfaces or thin lenses.

Note that $\mathbf{L}_{k+1}\mathbf{L}'_k$ is a product of matrices which belong to the same intermediate space. They are connected through a transfer and the product may be written as

$$\mathbf{L}_{k+1}\mathbf{L}'_k = \frac{(\mathbf{L}'_k)^2}{\mathbf{I} - t_k\mathbf{L}'_k} = \mathbf{L}'_k\mathbf{L}_{k+1} \tag{64}$$

This relation shows that the product of these 2 matrices is symmetric. This property follows also from the fact that the left-hand side of equation 63 is a symmetric matrix and the first sum on the right hand side as well. Then, the second sum has to produce a symmetric matrix as well.

The lateral magnification, relating the shifts in image and object space, can now be generalized to the following definition

$$\mathbf{M} = \frac{\mathbf{L}_K}{\mathbf{L}'_K}\frac{\mathbf{L}_{K-1}}{\mathbf{L}'_{K-1}}\cdots\frac{\mathbf{L}_1}{\mathbf{L}'_1} \tag{65}$$

which again reduces to the well-known equation in case of scalar quantities. If at least one of the inverse matrices does not exist, we have to deal with a singular case, which does not introduce a serious problem. The details are given in the appendix. Although this form of the lateral magnification recalls the scalar version, a different form may be more useful both in numerical calculations and in theoretical considerations. With the help of equation 60, we arrive at the expression

$$\mathbf{M} = \frac{1}{\mathbf{L}'_K}\frac{1}{\mathbf{I} - t_{K-1}(\mathbf{F}_{K-1} + \mathbf{L}_{K-1})} \times \ldots \times \frac{1}{\mathbf{I} - t_1(\mathbf{F}_1 + \mathbf{L}_1)}\mathbf{L}_1 \tag{66}$$

The general angular magnification, see equation 40, then reads

$$\mathbf{N} = \frac{1}{\mathbf{I} - t_{K-1}(\mathbf{F}_{K-1} + \mathbf{L}_{K-1})} \times \ldots \times \frac{1}{\mathbf{I} - t_1(\mathbf{F}_1 + \mathbf{L}_1)} \tag{67}$$

This expression is a product of factors each related to an intermediate space. A Taylor expansion for small reduced distances leads to the following approximation

$$\mathbf{N} \approx \mathbf{I} + \sum_{k=1}^{K-1} t_k(\mathbf{F}_k + \mathbf{L}_k) \tag{68}$$

which is a symmetric matrix. The order of the contributions to the sum plays no role. Only in cases in which this approximation fails, an asymmetric angular magnification might emerge. Seemingly, this case appears, when the linear Taylor approximation is not reliable enough to cope with nonlinear effects introduced by the free propagation of a wavefront in intermediate spaces.

In practical calculations, the inverse matrix $\mathbf{N}^{-1}$ will be calculated first according to

$$\mathbf{N}^{-1} = [\mathbf{I} - t_1\mathbf{L}'_1] \times [\mathbf{I} - t_2\mathbf{L}'_2] \times \cdots \times [\mathbf{I} - t_{K-1}\mathbf{L}'_{T-1}] \tag{69}$$

to minimize the number of matrix inversions. The reversed ordering should be recognized. To finish, the inverse of $\mathbf{N}^{-1}$ is calculated.

If a plane wave is generated in an intermediate space labeled k, the vergence $\mathbf{L}'_k$ is the null matrix and cannot be inverted. The vergence $\mathbf{L}_{k+1}$ after propagation through the intermediate space is the same plane wave and therefore $\mathbf{L}_{k+1} = \mathbf{L}'_k = \mathbf{0}$. Hence, the symbolic fraction $\mathbf{L}_{k+1} / \mathbf{L}'_k$ is indefinite. As can be seen from equation 67, the simple rule of cancelling equal factors in the nominator and denominator of a fraction may be applied, because the related factor in $\mathbf{N}$ just reduces to the identity matrix.

Another singular case has to be considered. If a focal line coincides with an interface, the straightforward application of the ASAM will cause numeric problems. Although the general mathematical treatment is postponed to the appendix, a fast workaround is at hand. We just simulate the process of calculating the limiting value. Therefore, we translate the problematic element by some atomic diameters, say $\varepsilon = 10^{-10}$m, along the optical axis. All programs based on a double precision representation of numbers will then return to an undisturbed computational flow. Obviously, any real-world results will not depend on the chosen value of ε, as long as it is small enough compared with the desired accuracy of the final result.

In the case of a distant object or a distant image, the matrices

$$\mathbf{M}_\infty = \frac{1}{\mathbf{L}'_K}\mathbf{N} \qquad {}_\infty\mathbf{M} = \mathbf{N}\mathbf{L}_1 \tag{70}$$

translate between angles and positions in object and image space and vice versa.

DISCUSSION

If a rotational symmetric system is considered in the paraxial approximation, the stepalong method for vergences provides us with the information required to characterize an optical system: the image distance and the magnification. By contrast, the linear optics approach to general systems produces all relevant information by evaluation of the ray transference. Hitherto the vergence approach based on the wavefront picture was incomplete and restricted to on-axis information only. The on-axis results are sufficient to model the effect of corrections like glasses, contact lenses, or intraocular lens by means of vergences but do not allow for magnification issues relevant, for example, for aniseikonia.

We took tilted wavefronts into account and showed that the missing information is already at hand, albeit hidden. Thus, the knowledge of the on-axis vergences produced by the stepalong method includes the off-axis behavior as well. The derived equations for the lateral and angular magnification are closely related to the expressions known from Gaussian optics. The difference is the fact that instead of scalar quantities, matrices have to be used. This calls for appropriate tools like matrix multiplication and matrix inversions. Because only 2×2 matrices are involved, the necessary toolbox is small.

The fundamental quantity regarding the off-axis behavior of the optical system under consideration is the angular magnification matrix $\mathbf{N}$. In a first step, the calculated vergences before and after each interface in the system are applied to calculate the inverse matrix $\mathbf{N}^{-1}$ given by equation 69. The inversion of this 2×2 matrix leads us to $\mathbf{N}$. The character of the system might be described by infinite conjugates (afocal system) by finite conjugates on each side or by only one finite conjugate. Depending on the situation, the matrix $\mathbf{N}$ can be augmented to yield the lateral magnification matrix $\mathbf{M}$ or the matrices ${}_\infty\mathbf{M}$ and $\mathbf{M}_\infty$ for an infinite image or object distance, respectively, as given in equation 70. Therefore, the ASAM provides all necessary information needed to describe a centered optical system in paraxial approximation.

Until now, we took only centered optical systems into account. Tilted and decentered elements can be included, as the section on Prentice's equation shows. A detailed investigation is underway.

The appendix is available online at www.optvissci.com

APPENDIX A

NUMERICAL EXAMPLE: PSEUDOPHAKIC EYE

As an example for demonstrating a numerical application, we choose the case of 3 interfaces, representing a pseudophakic eye with a spectacle correction. This example was introduced by Langenbucher et al.[15] All components, spectacle lens, cornea, and intraocular lens are treated as thin lenses with the following dioptric matrices (all units are diopters)

$$\mathbf{F}_{\text{spect}} = \begin{pmatrix} -2.2127 & 0.2972 \\ 0.2972 & -2.8282 \end{pmatrix}$$
$$\mathbf{F}_{\text{Cornea}} = \begin{pmatrix} 42.9970 & -1.2301 \\ -1.2301 & 45.5447 \end{pmatrix}$$
$$\mathbf{F}_{\text{IOL}} = \begin{pmatrix} 23.1586 & 1.3188 \\ 1.3188 & 20.4272 \end{pmatrix}$$

The reduced thicknesses are given by the following figures: the vertex distance of the spectacle $t_{\text{vert.}} = 0.014$m, the anterior chamber depth $t_{\text{ACD}} = 0.0035853$m. For the sake of completeness, the vitreous $t_{\text{vit}} = 0.0140793$m is given as well, although this number is not needed in the calculation. Assuming a distant object with $\mathbf{L}_1 = 0$, the following vergences emerge at the 3 interfaces, after repeated application of the relations for refraction, equation 59, and transfer, equation 60:

$$\mathbf{L}_1{}' = \mathbf{L}_1 + \mathbf{F}_{\text{spect}} = \begin{pmatrix} -2.2127 & 0.2972 \\ 0.2972 & -2.8282 \end{pmatrix}$$
$$\mathbf{L}_2 = \frac{\mathbf{L}_1{}'}{\mathbf{I} - t_{\text{vert.}}\mathbf{L}_1{}'} = \begin{pmatrix} -2.1451 & 0.2773 \\ 0.2773 & -2.7194 \end{pmatrix}$$
$$\mathbf{L}_2{}' = \mathbf{L}_2 + \mathbf{F}_{\text{Cornea}} = \begin{pmatrix} 40.8519 & -0.9528 \\ -0.9528 & 42.8253 \end{pmatrix}$$
$$\mathbf{L}_3 = \frac{\mathbf{L}_2{}'}{\mathbf{I} - t_{\text{ACD}}\mathbf{L}_2{}'} = \begin{pmatrix} 47.8674 & -1.3188 \\ -1.3188 & 50.5989 \end{pmatrix}$$
$$\mathbf{L}_3{}' = \mathbf{L}_3 + \mathbf{F}_{\text{IOL}} = \begin{pmatrix} 71.0260 & 0 \\ 0 & 71.0261 \end{pmatrix}$$

To calculate the magnification matrices, we start with

$$\mathbf{N}^{-1} = [\mathbf{I} - t_{\text{vert.}}\mathbf{L}'_1] \times [\mathbf{I} - t_{\text{ACD}}\mathbf{L}'2]$$

and arrive at the angular magnification

$$\mathbf{N} = (\mathbf{N}^{-1})^{-1} = \begin{pmatrix} 1.1364 & 0 \\ 0 & 1.1364 \end{pmatrix}$$

The matrix $\mathbf{M}_\infty$ for a distant object is calculated by

$$\mathbf{M}_\infty = \mathbf{L}_3{'}^{-1}\mathbf{N}$$
$$= \begin{pmatrix} 71.0260\text{ D} & 0\text{ D} \\ 0\text{ D} & 71.0261\text{ D} \end{pmatrix}^{-1} \begin{pmatrix} 1.1364 & 0 \\ 0 & 1.1364 \end{pmatrix}$$
$$= \begin{pmatrix} 0.0160\text{ m} & 0\text{ m} \\ 0\text{ m} & 0.0160\text{ m} \end{pmatrix}$$

in agreement with the figures presented by Langenbucher et al.[15]

APPENDIX B

DISCUSSION OF SINGULAR CASES

Although quite seldom in practical ophthalmic problems, there might exist cases, in which a focal line coincides with an interface where a refraction has to be carried out. Of course, a focal line can be imaged to a surface like the retina. However, this poses no problems at all. Only if the wavefront has to be traced further on and has to be refracted at that interface problems will show up.

At first sight, the propagation process shows a singular behavior if a focal line appears at an interface. A proper treatment of the refraction at that interface is not available in the ASAM, because the involved quantities tend to infinity. In the case of a spherical wavefront, the solution is simple: ignore the interface and propagate the wavefront as if the surface were not there. Actually, those surfaces approximately appear in technical instruments and are known as field lenses. In the case of astigmatic wavefronts, things are more complicated as has been shown by Harris.[14] He solved the problem with the help of the ray transference. By contrast, we will show how the solution can be generated in the wavefront picture. Although of interest from a mathematical point of view, these singularities are not really an obstacle to numerical calculations, because the interface can always be translated along the optical axis by a tiny amount. This procedure is confident as soon as it can be shown that the limiting process leads to a finite result. This will be done in the following.

If the quantity Δ in equation 17 tends to zero, a focal line problem appears. This calls for the following treatment. We will translate the interface by an amount ε and therefore keep all involved quantities finite. We continue the calculation and, in the final result, we will consider the limiting process and allow for $\varepsilon = 0$.

Consider a vergence $\mathbf{L}$ propagating a reduced distance $t_1-\varepsilon$ to reach an interface with a given dioptric power matrix $\mathbf{F}$. The incoming vergence at the interface is denoted by $\mathbf{L}^*$. After refraction, we get the usual result $\mathbf{L}' = \mathbf{F} + \mathbf{L}^*$. Then we propagate the wavefront by an amount $t_2 + \varepsilon$ to arrive at the final result $(\mathbf{L}')^*$. To relieve the whole procedure, we switch to a coordinate system in which the vergence matrix $\mathbf{L}$ is diagonal with the diagonal elements $L_1$and L_2. The dioptric power matrix has to be transformed into this coordinate system, where the symbol $\bar{\mathbf{F}}$ is used for the transformed dioptric power matrix.

Arbitrarily, we may assume that $L_1 = 1/t_1$. The propagation by $t_1-\varepsilon$ and the following refraction leads to

$$\mathbf{L}' = \begin{pmatrix} \frac{1}{\varepsilon} + \bar{F}_{11} & \bar{F}_{12} \\ \bar{F}_{12} & \frac{L_2}{1-(t_1-\varepsilon)} + \bar{F}_{22} \end{pmatrix}$$

To propagate the wavefront again, we have to evaluate the quantity $1/\Delta$. Including only the leading term in ε, we arrive at

$$\frac{1}{\Delta} \approx \frac{\varepsilon}{t_2}\left[\left(\frac{L_2}{1-(t_1-\varepsilon)} + \bar{F}_{22}\right)t_2 - 1\right]^{-1}$$

These results are inserted into

$$(\mathbf{L}')^* = \frac{1}{\Delta}\mathbf{L}' - \frac{t_2}{\Delta}(det\mathbf{L}')\,\mathbf{I}$$

Taking into account the limiting procedure $\varepsilon \to 0$ and some arithmetic transformations led us to the final and finite result

$$(\mathbf{L}')^* = \begin{pmatrix} (L_{11}')^* & 0 \\ 0 & (L_{22}')^* \end{pmatrix}$$

where

$$(L_{11}')^* = -\frac{1}{t_2}$$
$$(L_{22}')^* = \frac{L_2 + (1 - t_1L_2)\bar{F}_{22}}{1 - (t_1 + t_2)L_2 - t_2(1 - t_1L_2)\bar{F}_{22}}$$

The emerging vergence is still diagonal in the chosen reference coordinate system. The principal vergence producing the focal line problem is not affected by the interface similar to spherical wavefronts. The principal vergence along the focal line undergoes a change according to the curvital power along that line, which is given by $\bar{F}_{22}$. If there is no lens at all, we have $\bar{F}_{22} = 0$ and the expected propagation $L_2 / (1-(t_1 + t_2) L_2)$ shows up.

The same problem of infinite matrices will arise while calculating the magnification matrix. If a focal line appears at an interface, the computation will run into problems, because a factor will tend to infinity. The easiest way to tackle the problem is to look at equation 69 for the inverse angular magnification matrix. The appearance of a focal line at an interface will affect always 2 factors, say

$$[\mathbf{I} - t_k\mathbf{L}_k{'}]\cdot[\mathbf{I} - t_{k+1}\mathbf{L}_{k+1}{'}]$$

the first one will tend to zero and the second one will tend to infinity. In the coordinate system of the principal vergences, the matrices are diagonal. Let us assume that the upper left diagonal elements are the ones that might be indefinite. We apply the same limiting procedure as previously to the product of the 2 factors and one can show that the limit reads

$$[1 - t_k(L_k{'})_{11}][1 - t_{k+1}(L_{k+1}{'})_{11}] \to t_k/t_{k+1}$$

As expected, the final result shows no singular behavior.

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