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Comments on 'The average eye' by W.F. Harris, Ophthal. Physiol. Opt. 2004 24: 580–585

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The definition of an average eye is a non-trivial problem. Harris proposed a procedure based on ray transferences in the framework of linear optics. Because the symplectic transferences do not form a vector space transferences might be added but the result is not symplectic. In other words, the resulting matrix does not represent an optical system. Thus, the arithmetic average is not at hand.

Harris for the first time brought in a non-linear average of transferences, which seems to be a working solution for calculating averages. The starting point of Harris are the transferences of the optical systems to be averaged. In the case of averaging eyes they represent the properties of the considered eyes each taken as a whole. If the eye (or any other optical system) is taken as a whole, i.e. considering only the mapping rendered by the related transference, then the making of the transferences is a non-issue. How and by which components the transference was constructed is a question which is simply superfluous in this holistic approach. Harris coined it as follows: 'Similarly one can determine the average corneal power and the average of the properties of other components of the eye but such averages are averages for those particular parts of the eye and not of the optical character of the eye taken as a whole. Indeed, as we shall see, an "average eye" determined naïvely by averaging individual properties is, in general, not a possible eye at all'.

The last claim might be misleading as a detailed knowledge of the construction data (radii, thicknesses and refractive indices) allows for an alternative definition of an average eye as will be shown below. However, only if the supplementary information is available and only if the number of elements is the same for all eyes to be averaged is the proposed procedure feasible. Contrary to this restriction Harris' general approach can be applied in the moment the transferences of optical systems are at hand. The mean for a schematic eye with one surface only and, say, a Gullstrand eye with six surfaces, can easily be calculated. Even an average for binoculars, eyes and microscopes could be produced. Clearly, there might be little interest in such exotic combinations.

In the following it will be shown that an average eye determined by averaging individual properties is a possible eye. This procedure might be of interest while constructing model eyes from biometric data. Consider a set of biometric data for K eyes. These data comprise information on corneal and lens radii, refractive indices and thicknesses of the components. Ignoring tilts etc. the 4×4 system matrix (or transference) $\mathbf{S}$ for one of these eyes is given by a product of matrices. Because there are only two events in the life of a light ray each factor in this product is related either to a refraction ($\mathbf{R}$) or to a transfer ($\mathbf{T}$) which are given by

$$\mathbf{R} = \begin{pmatrix} \mathbf{I} & \mathbf{O} \\ -\mathbf{F} & \mathbf{I} \end{pmatrix} \qquad \mathbf{T} = \begin{pmatrix} \mathbf{I} & \frac{d}{n}\mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{pmatrix} \tag{1}$$

where $\mathbf{I}$ denotes the 2×2 identity matrix and $\mathbf{O}$ the 2×2 null matrix. The surface power matrix is given by $\mathbf{F}$ and the reduced thickness by d/n.

The following reasoning does not depend on the number of elements as long as all eyes to be averaged have the same number of elements. Therefore, we consider a typical example emphasising the fact that all construction data have to be at hand. If the considered eyes consist of a cornea and a simple eye lens the whole system matrix including the vitreous is made up by eight elements and reads

$$\mathbf{S} = \mathbf{T}_{\text{vit}} \underbrace{\mathbf{R}_{\text{L2}}\mathbf{T}_{\text{L}}\mathbf{R}_{\text{L1}}}_{\text{Lens}} \mathbf{T}_{\text{aqu}} \underbrace{\mathbf{R}_{\text{C2}}\mathbf{T}_{\text{C}}\mathbf{R}_{\text{C1}}}_{\text{Cornea}} \tag{2}$$

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and each event can be averaged, because the average matrix $\bar{\mathbf{R}}$ or $\bar{\mathbf{T}}$ is of the same type as its constituents, if the following definitions are used to calculate an average component from the biometric data

$$\bar{\mathbf{R}} = \sum_{k=1}^{K} w_k \mathbf{R}^{(k)} \qquad \bar{\mathbf{T}} = \sum_{k=1}^{K} w_k \mathbf{T}^{(k)} \tag{3}$$

with normalised weights

$$\sum_{k=1}^{K} w_k = 1 \tag{4}$$

These weights might represent sample sizes and other factors. Equivalently, the reduced thicknesses and the surface power matrices may be averaged and inserted in Equation (1). As a side issue it is worth mentioning that surface curvatures and not radii have to be averaged in this approach.

As can be shown easily, both averages $\bar{\mathbf{R}}$ and $\bar{\mathbf{T}}$ fulfill the symplectic relations

$$\bar{\mathbf{R}}'\mathbf{J}\bar{\mathbf{R}} = \left(\sum_{k=1}^{K} w_k \mathbf{R}^{(k)}\right)' \mathbf{J} \left(\sum_{k=1}^{K} w_k \mathbf{R}^{(k)}\right) = \mathbf{J} \tag{5}$$

$$\bar{\mathbf{T}}'\mathbf{J}\bar{\mathbf{T}} = \left(\sum_{k=1}^{K} w_k \mathbf{T}^{(k)}\right)' \mathbf{J} \left(\sum_{k=1}^{K} w_k \mathbf{T}^{(k)}\right) = \mathbf{J} \tag{6}$$

where

$$\mathbf{J} = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{I} & \mathbf{O} \end{pmatrix} \tag{7}$$

Although the applied arithmetic means lead to symplectic matrices this is not the case for an arbitrary sum of matrices. For example $\mathbf{R}_1 + \mathbf{R}_2$ is not a symplectic matrix. Clearly, even the restricted sets of elementary matrices of type $\mathbf{R}$ or $\mathbf{T}$ do not form a vector space concerning the addition of symplectic matrices.

As the product of symplectic matrices is symplectic as well, the following average can be defined according to our considered example

$$\bar{\mathbf{S}} = \bar{\mathbf{T}}_{\mathrm{vit}} \bar{\mathbf{R}}_{\mathrm{L2}} \bar{\mathbf{T}}_{\mathrm{L}} \bar{\mathbf{R}}_{\mathrm{L1}} \bar{\mathbf{T}}_{\mathrm{aqu}} \bar{\mathbf{R}}_{\mathrm{C2}} \bar{\mathbf{T}}_{\mathrm{C}} \bar{\mathbf{R}}_{\mathrm{C1}} \tag{8}$$

and this is a possible eye by construction. This kind of average can be applied whenever the biometric data are given and the number of elements are the same throughout the sample eyes.

To compare numerical results the same model eyes as used by Harris in Example 5 of his appendix are used. As there seems to be an numerical error in the data given by Harris, the numerical values of the transferences for the two model eyes are given as well by

$$\mathbf{S}_1 = \begin{pmatrix} -0.0927 & 0.0473 & 0.0156 & 0.0003 \\ 0.0458 & 0.0349 & 0.0003 & 0.0159 \\ -69.0509 & 3.6963 & 0.8428 & 0.0236 \\ 3.5700 & -61.1144 & 0.0236 & 0.8655 \end{pmatrix} \tag{9}$$

and

$$\mathbf{S}_2 = \begin{pmatrix} -0.0661 & -0.0304 & 0.0148 & 0.0000 \\ -0.0309 & 0.0620 & 0.0000 & 0.0149 \\ -71.7436 & -1.7598 & 0.8884 & 0.0011 \\ -1.8035 & -63.2372 & 0.0011 & 0.9046 \end{pmatrix} \tag{10}$$

Following the recipe of Harris the average of both matrices reads

$$\bar{\mathbf{S}}_{\mathrm{H}} = \exp\left[\frac{1}{2}(\ln\mathbf{S}_1 + \ln\mathbf{S}_2)\right] \tag{11}$$

where matrix-functions have to be applied leading to

$$\bar{\mathbf{S}}_{\mathrm{H}} = \begin{pmatrix} -0.0801 & 0.0078 & 0.0152 & 0.0001 \\ 0.0068 & 0.0478 & 0.0001 & 0.0154 \\ -70.4148 & 1.0216 & 0.8651 & 0.0109 \\ 0.9367 & -62.1993 & 0.0110 & 0.8846 \end{pmatrix} \tag{12}$$

Applying the naïve procedure described above yields the following average

$$\bar{\mathbf{S}}_{\mathrm{N}} = \begin{pmatrix} -0.0799 & 0.0065 & 0.0152 & 0.0001 \\ 0.0054 & 0.0476 & 0.0001 & 0.0154 \\ -70.3282 & 0.9649 & 0.8659 & 0.0122 \\ 0.8702 & -62.1354 & 0.0122 & 0.8855 \end{pmatrix} \tag{13}$$

The small difference between both averages is given by

$$\bar{\mathbf{S}}_{\mathrm{N}} - \bar{\mathbf{S}}_{\mathrm{H}} = \begin{pmatrix} 0.0002 & -0.0013 & 0.0000 & 0.0000 \\ -0.0014 & -0.0001 & 0.0000 & 0.0000 \\ 0.0866 & -0.0566 & 0.0008 & 0.0013 \\ -0.0665 & 0.0639 & 0.0012 & 0.0009 \end{pmatrix} \tag{14}$$

As we know from other averages (arithmetic, median etc.) the difference between various types of averages is small as long as the variance of the sample is small as well. However, if quite diverse data sets are averaged bigger differences will show up and have to be interpreted carefully.

A speculation will finish this comment. Up to now there exists no proof that Harris' average is always leading to a symplectic matrix. As any transference is a word (product) spelled by the elementary matrices $\mathbf{R}$ and $\mathbf{T}$, a proof might rest on this fact.