

# Modified point diffraction interferometer for inspection and evaluation of ophthalmic components

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We demonstrate that a modified point diffraction interferometer can be used to measure the power distribution of different kinds of ophthalmic lenses such as spectacles, rigid and soft contact lenses, progressive lenses, etc. The relationship between the shape of the fringes and the power characteristics of the component being tested is simple and makes the design a very convenient and robust tool for inspection or quality control. Some simulations based on the Fresnel approximation are included. © 2006 Optical Society of America

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## 1. INTRODUCTION

The trend in ophthalmic optics nowadays is to customize corrective solutions for users. Therefore, the vision optics industry needs fast, versatile, and accurate methods to characterize ophthalmic elements for production and research. Wavefront sensing technology based on interferometry has not been very successful for a number of reasons such as coherence requirements, restricted dynamic range, and the need for a reference beam, making it difficult to implement in a robust apparatus as required by commercial applications.

A point diffraction interferometer<sup>1</sup> (PDI) is a conceptually simple and inexpensive tool used to measure wavefronts. The PDI is a two-beam interferometer in which a spherical reference beam is produced by diffraction at a very small, clear pinhole in a semitransparent coating. In our view, a PDI is a very simple and efficient interferometer, but apart from its initial applications in testing astronomical telescopes,<sup>1</sup> extreme ultraviolet interferometry,<sup>2,3</sup> and analysis of flames,<sup>4</sup> it has not been exploited to any significant degree in other applications.

There are three features that make a PDI attractive for implementation in compact and robust devices for measuring ophthalmic components: the PDI is a common-path interferometer and is therefore not sensitive to vibration, it employs only a single element, and it has the capability of directly measuring optical path differences.

Figure 1(a) shows the basic scheme of a PDI for measuring phase objects. A semitransparent plate with a clear, circular pinhole is placed at the focal plane of a lens FL. If a collimated beam illuminates a transparent object TO, an aberrated plane wave will be produced and will be focused on the PDI plate by the lens FL. As a result of dif-

fraction, the small pinhole will generate a spherical beam that will interfere with the aberrated wavefront transmitted through the semitransparent region [Fig. 1(b)]. In an observing plane I, the intensity distribution will contain all the information about the aberrations produced by the transparent object and not only higher-order aberrations but tilt, defocus, and astigmatism (or prism, sphere, and cylinder as termed in ophthalmic optics).

To obtain a good interferogram only two factors in the design of the PDI plate have to be taken into account. First, the diameter of the pinhole must be smaller than half of the Airy disk for the pinhole to generate a good approximation to a spherical reference wave. Second, the transmission factor of the semitransparent coating has to be chosen carefully to achieve sufficient contrast since the transmission factor depends not only on the size of the pinhole but also on the type and amount of aberration.<sup>5</sup>

PDI's have always been designed to observe small phase changes, i.e., either to visualize very small phase changes produced by a phase object (such as a flame) or to measure the quality of an optical element, assuming that the element under test has only a small amount of aberration.<sup>6,7</sup> In this sense, it may be difficult to imagine that the output beam of a 4 D ophthalmic lens, for example, can be regarded as a plane wave with "some amount of defocus" and therefore it may be also difficult to imagine that it can be measured with a PDI—and this is, in fact, impossible!

## 2. HOLE DIFFRACTION INTERFEROMETER

For the purpose of measuring large amounts of defocus, we have developed an instrument working basically on

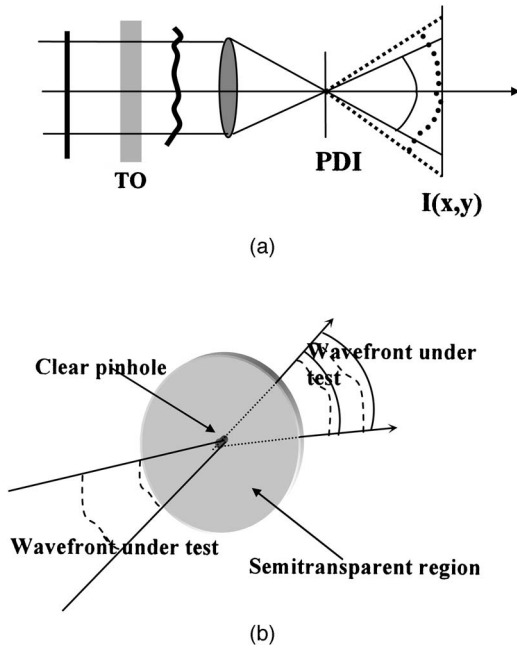


Fig. 1. (a) Basic scheme for the visualization of the phase produced by a transparent object (TO). (b) Principle of the PDI.

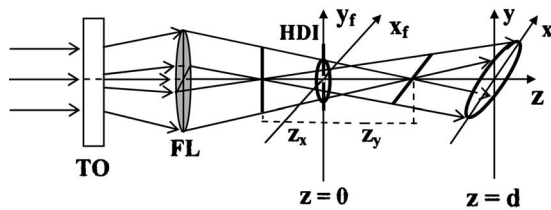


Fig. 2. Scheme for visualization of the phase produced by an ophthalmic lens.

the same principle as a PDI, but whose pinhole has a larger diameter than the Airy pattern of the unaberrated wavefront. We will refer to this as a hole diffraction interferometer (HDI). This interferometer can be considered rudimentary in terms of detail, but in return it can accurately resolve large amounts of aberration. In other words, increasing the pinhole size enhances dynamic range at the expense of sensitivity.

In order to study the performance of the HDI we have carried out numerical simulations to analyze the behavior of the interference pattern for large amounts of defocus and astigmatism. Theories and simulations have been made in previous works<sup>5,7</sup> but only for very tiny pinholes. Here, we took into account not only the diameter of the hole and the transmittance of the coating but also the phase that the semitransparent coating may add to the aberrated wavefront and the Gouy phase shifts. To our knowledge this is the first time that these elements have ever been included. Nevertheless, the consideration of the different phases is important in that it allows for the design of a HDI capable of distinguishing among positive and negative defocus in a still picture.

Let us assume that a transparent object (for instance, an ophthalmic lens) induces a given amount of sphere and cylinder in the collimated beam; i.e., a wavefront with astigmatism is produced (see Fig. 2). The complex ampli-

tude distribution  $U(x_f, y_f)$  at the focal plane ( $z=0$ ) of the focusing lens FL will be given by

$$U(x_f, y_f) \propto P(x_f, y_f) \exp \left[ \frac{ik}{2} \left( \frac{x_f^2}{z_x} + \frac{y_f^2}{z_y} \right) \right], \quad (1)$$

$z_x$  and  $z_y$  being the distances from the focal plane to the focal lines of the emerging beam and

$$P(x_f, y_f) = \begin{cases} 1 & \text{if } (x_f, y_f) \text{ lies inside the geometrically lit} \\ & \text{region at } z = 0 \\ 0 & \text{elsewhere} \end{cases}.$$

Therefore, the complex amplitude distribution according to the Fresnel approach at the observing plane ( $z=d$ ) will be given by

$$\begin{aligned} U(x, y) &\propto [1 - t \exp(i\Phi)] \int_{\Sigma} \exp \left[ \frac{ik}{2} \left( \frac{x_f^2}{z_x} + \frac{y_f^2}{z_y} \right) \right] \\ &\times \exp \left\{ \frac{ik}{2d} [(x - x_f)^2 + (y - y_f)^2] \right\} dx_f dy_f + t \exp(i\Phi) \\ &\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(x_f, y_f) \exp \left[ \frac{ik}{2} \left( \frac{x_f^2}{z_x} + \frac{y_f^2}{z_y} \right) \right] \\ &\times \exp \left\{ \frac{ik}{2d} [(x - x_f)^2 + (y - y_f)^2] \right\} dx_f dy_f, \end{aligned} \quad (2)$$

with  $\Sigma$  denoting the hole area, i.e., a circular aperture with radius  $a$ ;  $t$  the transmission coefficient of the semitransparent mask;  $\Phi$  the phase that the semitransparent mask may add to the nondiffracted wave due, for instance, to the thickness of the metallic coating; and  $d$  the distance from the focal plane to the observing plane. (All the factors common to both integrals have been left out of the formula.)

Assuming  $a \ll z_x, z_y \ll d$ , the first sum in the right-hand side of relation (2) reduces to<sup>8</sup>

$$[1 - t \exp(i\Phi)] a^2 \frac{J_1[(akr)/d]}{[(akr)/d]}, \quad (3)$$

where  $r = (x^2 + y^2)^{1/2}$  and  $J_1$  represents the Bessel function of the first kind.

The second integral represents the complex amplitude distribution of the beam emerging from the lens FL at the observing plane at  $z=d$ . Since the observing plane is distant from the vicinities of the focal plane of FL and the focal lines generated by the astigmatic ophthalmic component (or from caustics regions in general), then the diffraction effects due to the finite size of the FL will, in practice, be negligible so integration can be asymptotically evaluated by the method of stationary phase by taking into account only the contribution of the critical points of the first kind.<sup>9,10</sup> Although this is a rather heuristic approach, it nevertheless produces accurate results that are confirmed experimentally. Thus, the second integration produces

$$P(x,y)\exp(i\Phi)\exp(i\sigma t) \frac{d \exp \left[ \frac{ik}{2d} \left( \frac{z_x x^2}{(d+z_x)} + \frac{z_y y^2}{(d+z_y)} \right) \right]}{k \left[ \left| \frac{(d+z_x)(d+z_y)}{z_x z_y} \right| \right]^{1/2}}, \quad (4)$$

where

$$P(x,y) = \begin{cases} 1 & \text{if } (x,y) \text{ lies inside the geometrically lit} \\ & \text{region at } z = d \\ 0 & \text{elsewhere} \end{cases},$$

and

$$\sigma = \begin{cases} \pi/2 & \text{if } z_x z_y > 0 \text{ and } z_x > 0 \\ 3\pi/2 & \text{if } z_x z_y < 0 \text{ and } z_x < 0 \\ \pi & \text{if } z_x z_y < 0 \end{cases}.$$

$\sigma$  takes into account the Gouy effect, i.e., the phase shift that the converging beam undergoes as it passes through both focal lines. (Particular cases where  $z_x=0$  or  $z_y=0$  have to be treated separately, but the qualitative final results are equivalent; therefore we do not include them here.)

To get interference fringes with good contrast the transmission coefficient  $t$  must be very small, because the intensity transmitted through the pinhole is directly related to the encircled energy, so in relation (3) we can make the approximation  $[1 - t \exp(i\Phi)] \approx 1$ . Thus at the observing plane

$$I(x,y) \propto a^4 \frac{J_1^2[(akr)/d]}{[(akr)/d]^2} + \frac{t^2 d^2 P(x,y)}{k^2 \left| \frac{(d+z_x)(d+z_y)}{z_x z_y} \right|} + 2a^2 \frac{J_1\left(\frac{akr}{d}\right)}{\left(\frac{akr}{d}\right)} \frac{tdP(x,y)}{k \left[ \left| \frac{(d+z_x)(d+z_y)}{z_x z_y} \right| \right]^{1/2}} \times \cos \left\{ \frac{k}{2d} \left[ \frac{z_x x^2}{(d+z_x)} + \frac{z_y y^2}{(d+z_y)} \right] + \Phi + \sigma \right\}. \quad (5)$$

Figures 3 and 4 show the simulation in gray scale of several interference patterns described by relation (5). In all of them the diameter of the pinhole was  $14 \mu\text{m}$ , the HDI-observation plane distance was 5 cm and the geometrically lit region  $P(x,y)$  was assumed to be larger than the printed area; the amplitude-related transmission coefficient of the coating was  $t=0.1$  and the wavelength was  $0.633 \mu\text{m}$ . It was assumed that the coating of the semi-transparent mask adds a global phase of  $\Phi=\pi/2$  to the nondiffracted wave. In this way it is possible to obtain a central maximum for  $z_x > 0$  and  $z_y > 0$  (Fig. 3) or a central

minimum for  $z_x < 0$  and  $z_y < 0$  [Fig. 4(a)]; therefore the HDI is able to discriminate among positive or negative values of  $z_x$  and  $z_y$  (needless to say, if  $\Phi=3\pi/2$  the reverse situation could be obtained). Thus, if the ophthalmic component is a spherical lens then  $z_x=z_y=z>0$  (hereafter this situation will be referred to as positive defocus) and the interference pattern consists of circular fringes that have a maximum in the center, in contrast with a negative defocus  $z<0$  in which the fringes have a minimum. If the lens is astigmatic there will be two focal lines focused either before the HDI plate,  $z_x<0, z_y<0$  [elliptic fringes with a minimum in the center, as in Fig. 4(a)] or after the HDI plate,  $z_x>0, z_y>0$  (elliptic fringes with a maximum in the center); or one focal line after and one focal before the HDI plate,  $z_x<0, z_y>0$  (hyperbolas that present a saddle point in the center, as in Fig. 4(b)). Nevertheless, in this last case it is still possible to identify the position of the focal lines relative to the HDI plate by observing the direction in which the first absolute minimum (maximum) appears because this means that the focal line in the same direction is placed before (after) the HDI mask.

Figure 5 shows the intensity plots in the transverse direction  $x$  corresponding to interference patterns of Fig. 3

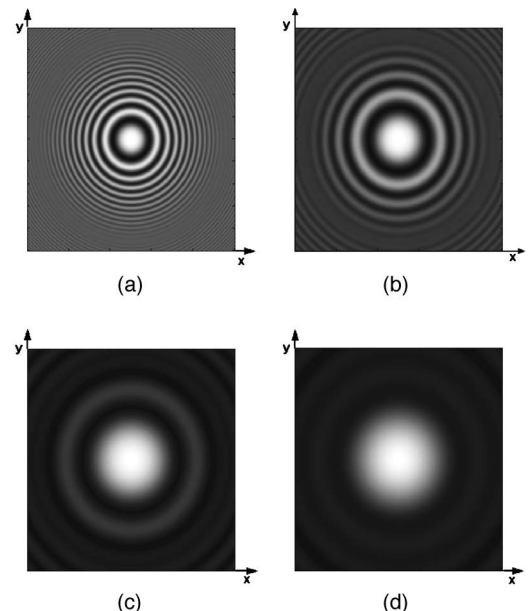


Fig. 3. Simulated interferograms for different amounts of defocus:  $z$  (in mm) = (a) 5, (b) 2, (c) 0.8, (d) 0.5. In all cases  $a=7 \mu\text{m}$ ,  $t=0.1$ ,  $d=5 \text{ cm}$ ,  $\Phi=\pi/2$ .

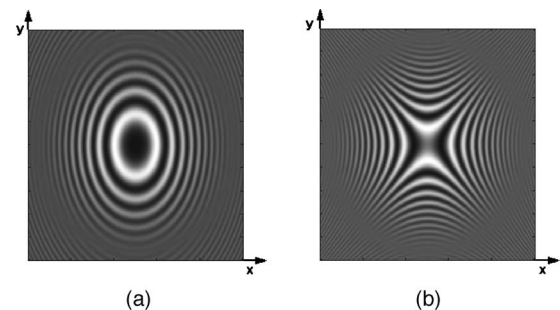


Fig. 4. Simulated interferograms of astigmatic wavefronts: (a)  $z_x=-5 \text{ mm}$ ,  $z_y=-2.5 \text{ mm}$ ; (b)  $z_x=5 \text{ mm}$ ,  $z_y=-2.5 \text{ mm}$ . In all cases  $a=7 \mu\text{m}$ ,  $t=0.1$ ,  $d=5 \text{ cm}$ ,  $\Phi=\pi/2$ .

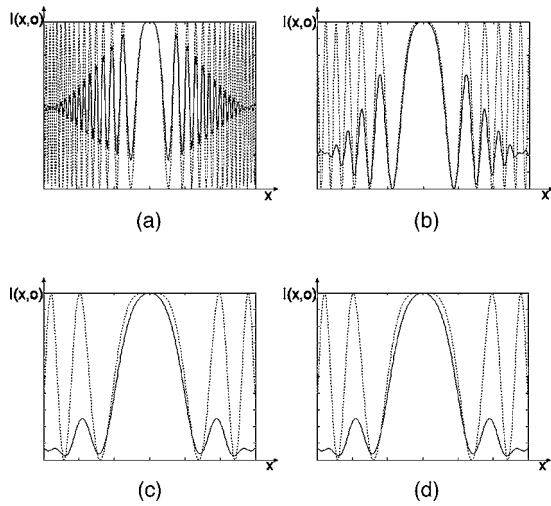


Fig. 5. Intensity plots in the  $x$  direction for the simulated interferograms of Fig. 3 (solid curves) and in the corresponding “ideal intensity plots” (dashed curves):  $z$  (in mm) = (a) 5, (b) 2, (c) 0.8, (d) 0.5.

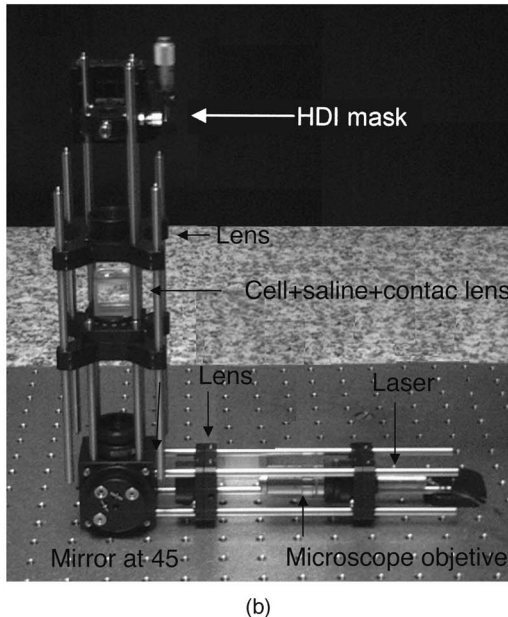
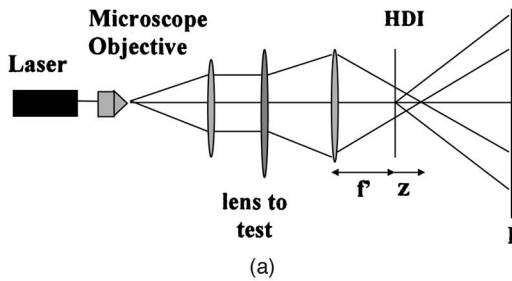


Fig. 6. (a) Schematic of the simplest experimental setup. (b) Photograph of the experimental device.

(solid curves); they are to be compared with dashed curves resulting from the interference pattern at  $z=d$  produced by a point source at  $z=0$  and an astigmatic beam, the focal lines of which are  $z_x$  and  $z_y$  from  $z=0$ , both beams having the same uniform amplitude at  $z=d$  and  $\Phi=\sigma=0$ , i.e.,

$$I(x,y) \propto 1 + 1 + 2 \cos \left\{ \frac{k}{2d} \left[ \frac{z_x x^2}{(d+z_x)} + \frac{z_y y^2}{(d+z_y)} \right] \right\}. \quad (6)$$

The latter are referred to as “ideal intensity plots.”

It can be deduced first that the effect of the pinhole size is to apodize the interference fringes. For large values of  $a$  the factor  $J_1(v)/v$  cannot be regarded or approached as a constant within the observing region, and this has two consequences:

1. For large amounts of aberration (large values of  $z_x$  and  $z_y$ ), a significant number of fringes will appear within the observing region and the effect of the pinhole size translates into a loss of visibility at the periphery, but the positions of maxima and minima do not change with regard to the positions of extrema of relation (6). Nevertheless, the number of visible fringes will be sufficient to determine accurately the values of the second-order aberrations.

2. For small amounts of aberration the effect of losing visibility at the periphery is obvious since there will be a low number of fringes. Additionally, the intensity distribution due to the Bessel function masks the positions of maxima and minima.

Here it should be stressed that by reducing the pinhole diameter and seeking an appropriate value of the transmission coefficient  $t$  (in order to obtain well-contrasted interferograms<sup>5</sup>) smaller amounts of aberration can, of course, be detected. At the limit, for pinholes with diameters that allow detection of a few wavelengths, the HDI tends to be a PDI. Nevertheless, very small holes may, from a theoretical point of view, solve very large amounts of aberration but at the expense of dramatically reducing the transmission coefficient because (a) the spherical beam generated by the pinhole has an amplitude directly related to the area of the pinhole, and (b) the encircled energy within the pinhole diminishes as the amount of aberration increases, since the geometrically lit region increases in the plane of the HDI mask.

So, for a given amount of aberration it is possible to find the optimum pair of values of  $t$  and  $a$  that maximizes the contrast of the interferograms. Each specific application of the HDI (amount and kind of aberration to be measured) requires a separate analysis to find the best ratio  $t:a$  and to optimize the performance of the interferometer.

Figure 6(a) represents one of the simplest schemes for the experimental setup of all the very many possible variations that can be imagined. We followed this scheme in the laboratory in a vertical configuration because this allowed us to obtain the interferograms of soft contact lenses immersed in a glass cell filled with saline solution [see Fig. 6(b)]. A laser diode module is focused by a microscope objective and collimation is produced with a lens. The lenses being tested alter the collimation of the beam that converges in the vicinity of the image focal plane of the second lens FL where the HDI mask is placed. The interference pattern at the observing plane  $I(x,y)$  is directly related to the axis, sphere, cylinder, and higher-order aberrations of the lens being tested.

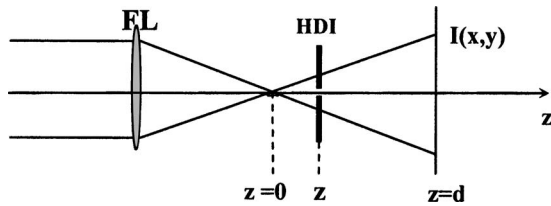
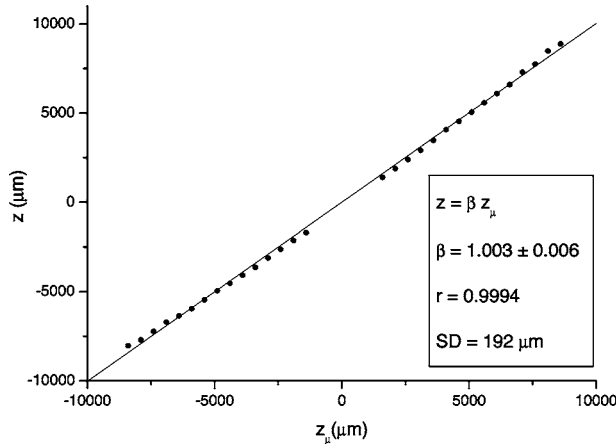


Fig. 7. Basic scheme for calibration of the interferometer.

Fig. 8. Fit for calibration of the experimental setup:  $f^* = 2.5$  mm,  $NA = 0.25$ ,  $t = 0.07$ ,  $\lambda = 0.633$   $\mu\text{m}$ .**Table 1. Range of Diopters That Can Be Solved by the HDI for Different Focusing Lenses and Pinhole Diameters<sup>a</sup>**

| FL<br>$f^*$ (cm) | Pinhole Diameter<br>( $\mu\text{m}$ ) | Dynamic Range<br>(D)<br>(Absolute Value) | Accuracy<br>(D) |
|------------------|---------------------------------------|------------------------------------------|-----------------|
| 5.0              | 7                                     | 0.20–1.60                                | 0.02            |
|                  | 10                                    | 0.25–3.50                                | 0.08            |
|                  | 15                                    | 0.60–5.00                                | 0.10            |
| 2.5              | 7                                     | 1.20–5.50                                | 0.01            |
|                  | 10                                    | 1.50–9.50                                | 0.04            |
|                  | 15                                    | 2.50–13.50                               | 0.08            |

<sup>a</sup>In all cases the transmittance for the HDI mask was  $t = 0.07$ ;  $\lambda = 0.633$   $\mu\text{m}$ ;  $NA$  of FL = 0.25.

### 3. CALIBRATION OF THE HOLE-DIFFRACTION INTERFEROMETER

The quantitative performance of the device was determined using the following calibration scheme. The focusing lens FL was illuminated with a collimated wave. The focal plane of the lens was chosen as the origin of coordinates ( $z = 0$ ), and the HDI mask, which was initially placed at the focal plane, was subsequently moved along the axis with a micrometric stage, thus simulating different amounts of defocus (see Fig. 7). For each axial displacement, the radial positions of the maxima and minima of the interferograms  $r_m^2$  were fitted to the corresponding fringe number  $m$  ( $m = 0$  at the center) and the value of the defocus obtained from the fit coefficient, since

$$m \approx \frac{|z| r_m^2}{\lambda d^2} \quad (7)$$

for  $|z| \ll d$ ,  $d$  being the distance from the focal plane of the

lens to the CCD camera and  $\lambda$  the wavelength used. The HDI mask was made by lithographic techniques in a Cr coating of a glass substrate. Its optical density was about 2.3. Due to the Cr coating the interferograms corresponding to displacement toward the observing plane presented a central minimum, and toward the focusing lens a central maximum. Therefore, for this coating we can consider  $\Phi = \pi/2$ . (Here it must be noted that we found that Al coating reverses the contrast of the interferograms compared with Cr coating, i.e., we can consider that  $\Phi = 3\pi/2$  for Al coating.)

Figure 8 shows the results of the calibration for a focusing lens FL with a focal length of 2.5 cm ( $NA = 0.25$ ) and a hole with a diameter of 15  $\mu\text{m}$ . The displacement of the mask measured with a micrometer ( $z_\mu$ ) is plotted against the defocus ( $z$ ) obtained from the fit of the interferogram data to relation (7). As can be observed, there are no measurements near the origin since, as explained above, a large hole cannot resolve a small amount of defocus. Displacement larger than 8500  $\mu\text{m}$  could not be performed because of limitation in the travel of the translation stage.

We must point out here that fringes were manually traced and the stage used to displace the mask had some limitations in terms of travel (as noted) and accuracy; therefore measurement errors can be smaller than those presented in this work. On the other hand, this calibration can be used to briefly and simply illustrate the dynamic range and accuracy (within our laboratory constraints) of the specific assembly of Fig. 6 in accordance with the following reasoning: One could postulate that a given amount of defocus  $z$  is produced by a spherical lens placed at the object focal plane of FL with power

$$P \approx \frac{-z}{f^{*2}} \text{ (D)} \quad (8)$$

(where Newton's equation has been used). Thus, from the data of Fig. 7 it can be deduced that this specific setup [FL with a focal length of 2.5 cm ( $NA = 0.25$ ) and a hole of diameter 15  $\mu\text{m}$ ] could measure ophthalmic spectacles from  $-13.50$  D to  $-2.50$  D and from  $+2.50$  D to  $13.50$  D [by taking into account the standard deviation (SD) of the fit we can ensure an accuracy of  $\approx 0.08$  D]. Table 1 lists the dynamic ranges as well as the accuracies we have obtained by using different FL and hole diameters. In all cases, the limits of the measurable diopters can be extended by performing small changes in the setup; the accuracy can also be increased by improvements in the mask manufacturing process, in the image processing of fringes, and in the quality of the stages.

### 4. EXPERIMENTAL RESULTS

Figures 9–11 show interferograms for different ophthalmic components. These images were obtained using several HDI masks manufactured on glass substrates with Cr and Al coatings, optical densities ranging from 2 to 3, and pinholes ranging from 7 to 15  $\mu\text{m}$ , for focusing lenses FL with  $f/2$  and  $f/4$ . Soft contact lenses were immersed in a glass cell containing saline solution. As the main aim of this work is to show the potential use of the device for the characterization of ophthalmic components,

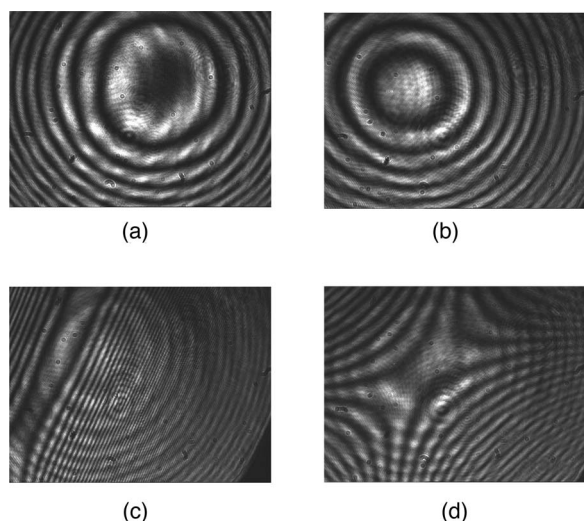


Fig. 9. Interferogram for different spectacle lenses. Displacements from the center are due to displacements of the lenses as well as prism effects. (a) positive sphere, (b) negative sphere, (c) positive cylinder, (d) toric.

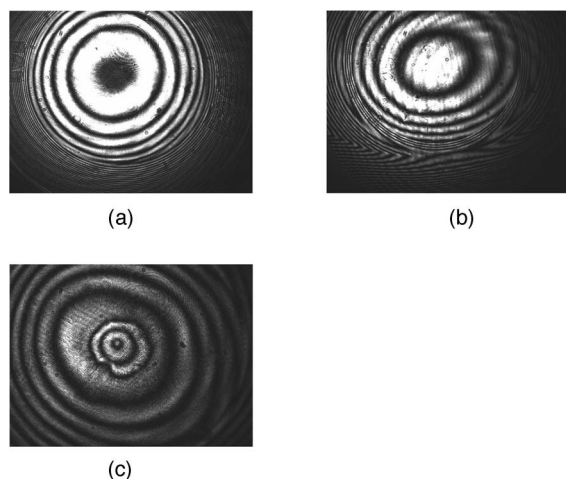


Fig. 10. Interferogram for different soft contact lenses in saline solution. Displacements from the center are due to displacements of the lenses as well as prism effects. (a) positive sphere, (b) toric, (c) dehydrated (after one hour exposed to air).

we do not give the details of the HDI masks and configurations we have used for each specific application of the device. In fact, not all the design parameters have been optimized for each specific purpose.

## 5. CONCLUSION

We have demonstrated the possibility of using a modified PDI for testing different ophthalmic components. The modification simply entails increasing the pinhole size beyond the Airy pattern diameter of the unaberrated wavefront. In this way the interferometer loses sensitivity in the detection of small amounts of aberration but its dynamic range is increased, providing interferograms with a reasonable degree of contrast for large amounts of aberration. The resulting interferometer, termed a hole diffraction interferometer (HDI), may be used as an accurate, inexpensive, and handy tool for the measurement of the optical characteristics of ophthalmic optics.

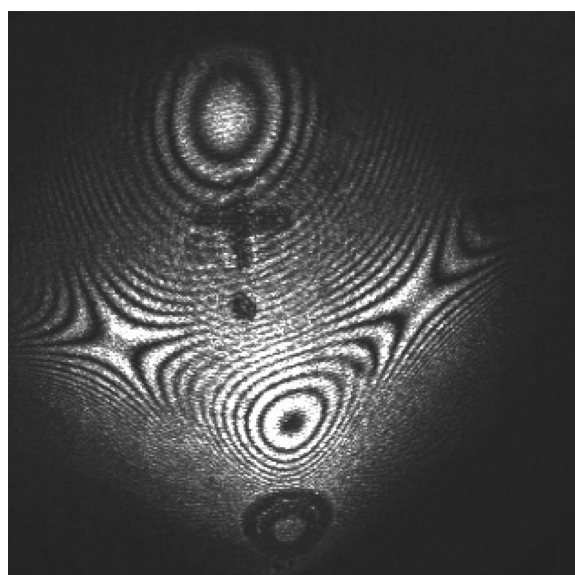


Fig. 11. Progressive lens. Note the change of sign in defocus from the upper part (far zone) to the lower part (near zone). Astigmatism near the progressive corridor is clearly visible.

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