Simple approach to the generalized Minkwitz theorem

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The Minkwitz theorem plays an important role in the design of progressive addition lenses. Recently, this theorem has been generalized by Esser *e*t al. [J. Opt. Soc. Am. A 34, 441 (2017)] to non-umbilic lines under the assumption of a symmetric surface. We present a simplified derivation and generalize their findings to arbitrary but sufficient smooth surfaces.

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1. INTRODUCTION

Most presbyopic people use progressive addition lenses as an aid, if they are to see clearly over a range of object distances. These distances are encoded by spatial positions on the surfaces of such lenses. The canonical design for a progressive addition lens includes a far zone, a near zone and a connecting narrow corridor, which is used for intermediate distances.

In this progression zone the optical power increases smoothly and each tiny patch corresponds to a different object distance. Therefore, looking at a specific object point requires a combination of the eyes' gaze direction together with an appropriate head movement. In a way, the muscles of the neck take over the function of the ciliary muscle.

Although the recent freeform surface technology is able to generate almost every geometric surface shape, not every optical power distribution across the the surface is feasible. Constraints from differential geometry effectively limit the design possibilities.

The Minkwitz theorem informs us about an important constraint: if the power increases along an umbilic line (a line with equal principal curvatures), the astigmatism (the difference of the principal curvatures) increases twice as fast in the lateral direction normal to this line. Because this line constitutes the spine of the design, it is called the principal line, see Figure 1.

Given a tolerance on visual acceptable astigmatism, the progression zone only provides a restricted lateral width for undisturbed vision. Hence, the lateral field of view is substantially limited in the region of the corridor, because visual acuity is considerably reduced in the lateral direction.

Even in the most recent designs of progressive addition lenses this constraint is the most constricting one for the user of such lenses. Obviously, a better understanding of the Minkwitz theorem is helpful to tweak designs toward the optimum.

Recently, Esser *et al.* presented a generalized version of this theorem [1] . The authors abandoned the assumption of an umbilic line, and they demonstrated that the lateral astigmatism not only depends on the change rate of power, but also on the change rate of astigmatism along the principal line. However, the authors kept the assumption of a symmetric surface, with the principal line as a symmetry line. An exhaustive list of references is given in their publication, to which the interested reader is referred.

The authors derived their findings by application of straight forward, but lengthy methods from differential geometry. They paved a way to a much shorter and simpler derivation of this theorem, which will be presented in the next section. The assumption of a symmetric surface can be skipped along this way.

2. DERIVATION OF THE GENERALIZED MINKWITZ THEOREM

We consider a sufficient smooth surface and a given line on this surface, which will be denoted principle line. Along this line the design characteristics of the surface, including the change rate of curvature, are prescribed.

Around a point P, located on the principle line, we choose an infinitesimal small patch and introduce the following coordinate system: the z-axis parallels the surface normal in P; the y-axis is chosen tangential to the principle line, and the x-axis completes the right handed orthogonal coordinate system pointing into the lateral direction.

The elevation of the surface $f(x,y)$ in this small patch around P can be described by a function of the local coordinates x, y restricted to second order contributions:

$$f_{(2)}(x,y) = \frac{1}{2}\left(H(x^2+y^2) + A^0(x^2-y^2) + 2A^{45}xy\right) \quad (1)$$

Because of the chosen coordinate system no constant or linear terms appear in this equation. The functional form of Eq. (1) offers a simple interpretation in terms of the curvature-related optical properties of the surface. The coefficients represent the mean curvature (H), the aligned astigmatism along the coordinate axes (A^0), and the the oblique astigmatism along the bisecting lines (A^{45}).

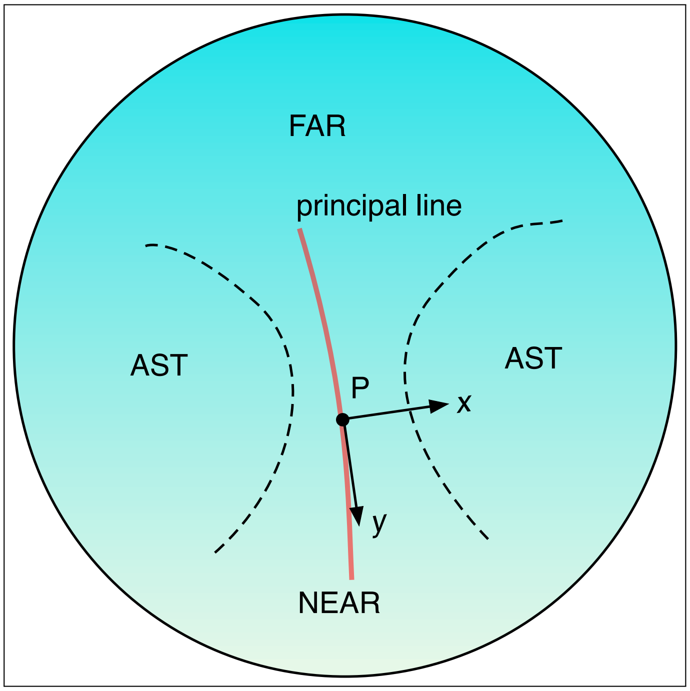

Fig. 1. Typical layout of a progression addition lens with three functional parts: the region for distant objects (top), the one for near objects (bottom), and the corridor for intermediate distances, where the power is changing along the principal line. The corridor is bordered by regions with high amounts of astigmatism. At a given point P a local coordinate system is chosen, such that the normal to the surface parallels the z-axis.

In terms of the coefficients of the second fundamental form, usually denoted L, M, N, we have

$$H = \frac{1}{2}(L+N) \quad (2)$$

$$A^0 = \frac{1}{2}(L-N) \quad (3)$$

$$A^{45} = M \quad (4)$$

The coefficients H, A^0, and A^{45} are constant for a given patch, but generally vary with its position and can be considered as functions of the position of the point P: $H(P), A^0(P)$ and $A^{45}(P)$. The functional form of Eq. (1) is the same at every point P, as long as the specific coordinate system described above is chosen.

We assume that the explicit global form of the surface $f(x,y)$, is given with respect to the specific coordinate system at a point P. Then the coefficients L, M, N can be calculated as functions of the coordinates x, y from the equations

$$L(x,y) = \frac{f_{xx}}{\sqrt{D}}$$

$$M(x,y) = \frac{f_{xy}}{\sqrt{D}}$$

$$N(x,y) = \frac{f_{yy}}{\sqrt{D}} \quad (5)$$

where

$$D = 1 + (f_x)^2 + (f_y)^2 \quad (6)$$

Partial derivatives are denoted by indices, e.g. $f_x = \partial f/\partial x$. It is assumed that the order of derivatives can be interchanged.

At each point, say $x = \bar{x}$ and $y = \bar{y}$, the values of L, M, N can be plugged into Eq. (1) to get the quadratic approximation at the point $\bar{x}, \bar{y}$. Note however, that in Eq. (1) local coordinates with respect to the specific coordinate system centered at $\bar{x}, \bar{y}$ have to be applied.

Of interest here are only infinitesimal changes of the coefficients L, M, N in the infinitesimal neighborhood of a point P. They can be calculated from equations (5) and are determined by the third order partial derivatives f_{xxx}, f_{xxy} etc., evaluated at the origin of the coordinate system $x = y = 0$. The contributions from the denominator, resulting from the product and chain rule, eventually collapse to a factor 1, since $f_x(0,0) = f_y(0,0) = 0$. We thus have $L_x = f_{xxx}$, etc., without any further contributions of first or second order derivatives. This is a specific property of the coefficients L, M, N resulting from the specific expressions in Eq. (5). In the general case, the fundamental properties mean curvature and Gaussian curvature both depend on L, M, N and on the coefficients of the first fundamental form E, F, G. As a reasoning along the same line as above shows, these coefficients do not generate additional contributions.

The design of the lens prescribes the progression of the mean curvature along the principle line. Roughly speaking, this is the difference between the optical power in the far and in the near zone divided by the length of the progression zone.

The partial derivative tangential to the principal line is a given function along that line. At a given point we have

$$H_y = \frac{1}{2}(L_y + N_y) = \frac{1}{2}(f_{xxy} + f_{yyy}) \quad (7)$$

The remaining first order partial derivatives of the coefficients are

$$A^{45}_x = M_x = f_{xyx} \quad (8)$$

$$A^0_y = \frac{1}{2}(L_y - N_y) = \frac{1}{2}(f_{xxy} - f_{yyy}) \quad (9)$$

$$A^{45}_y = M_y = f_{xyy} \quad (10)$$

$$A^0_x = \frac{1}{2}(L_x - N_x) = \frac{1}{2}(f_{xxx} - f_{yyx}) \quad (11)$$

$$H_x = \frac{1}{2}(L_x + N_x) = \frac{1}{2}(f_{xxx} + f_{yyx}) \quad (12)$$

The last three partial derivatives are zero for a symmetric surface because all odd numbered derivatives with respect to x are zero in this case.

Independent of the symmetry assumption, the combination of the first three equations, Eq. (7) -(9), renders the main and general result

$$A^{45}_x = H_y + A^0_y \quad (13)$$

which is equivalent to Eq. (29) of [1], where the following substitutions have to be applied to their notation: $\beta \to x$, $s \to y$, $(k + \Delta k/2)' \to H_y$, and $(\Delta k/2)' \to A^0_y$. Note, that the assumption of symmetry has been explicitly exploited in their derivation.

The combination of the remaining three equations, Eq. (10) - (12), leads to the lateral change of the aligned astigmatic component

$$A^0_x = H_x - A^{45}_y \quad (14)$$

A lateral change of the mean curvature H_x induces a change of magnification and is avoided in modern lenses, because a horizontal head turn would introduce unusual visual "breathing" effects in the image. Therefore, in real lenses, we often have $H_x \approx 0$. There is no advantage of introducing $A^{45}_y \neq 0$. So, the relation $A_x^0 \approx 0$ holds as well. Hence, actually produced lenses usually show nearly symmetric surfaces.

The equations (13) and (14) eventually reduce to the compatibility equations in their simplest form:

$$L_y = M_x \qquad M_y = N_x \tag{15}$$

These equations demand, that the third order mixed partial derivatives have to be independent of the order of differentiation. Therefore, the coefficients L, M, N can not be arbitrarily prescribed across a surface. Instead, the compatibility equations have to be fulfilled. When the design of a principal line is prescribed, the compatibility equations introduce constraints between changes of mean curvature and the both forms of astigmatism. The first constraint has been described by the Minkwitz theorem, which is by far the most important aspect from the perspective of practical applications in progressive addition lenses.

For the sake of simplicity, we omitted the factor $n' - n$, accounting for the difference between the refractive indices of the media behind and in front of the surface. This factor transforms curvatures into paraxial optical powers. In reality, rays or wave fronts are not confined to the paraxial region. Additionally, we have to consider the refraction at the second surface including the transfer to this surface. All these effects have to be considered when the whole progressive addition lens is taken into account. These calculations may be realized by numerical ray tracing, but it is difficult to describe them analytically. Therefore, we stay with the most simple properties of one surface only.

3. DISCUSSION

The generalized Minkwitz theorem, Eq. (13), can be derived quite easily for the general case, if an appropriate coordinate system in the tangential plane to a considered point P is chosen. The findings are then restricted to infinitesimal first order changes, which is appropriate for the derived result. The derivation by Esser *et al.* presents add-on insights, because higher order terms and their behavior are explicitly considered and discussed.

From a standpoint of optical design the generalized Minkwitz theorem introduces a new degree of freedom for the balancing of aberrations: the change rate of the coefficient A^0 along the principal line. The width of the progression corridor, and therefore the field of view, can be augmented, if the change rate A_y^0 is proportional to the negative rate of the mean curvature , $A_y^0 \propto -M_y$. Then, the oblique astigmatism increases with a slower rate than in the case of an umbilic line.

Vice versa, the theorem likewise states that the lateral change rate of A^{45} can only be traded off against the tangential change rate of A_y^0. It is not possible to avoid both of them.

Often, a reduced visual acuity due to astigmatism in the near zone is more acceptable than a reduced lateral field of view. This may be important in the case of wearing progressive addition lenses in front of computer monitors. Here, an increasing astigmatism supplemented by a reduced power change rate might be acceptable to widen the field of view in the corridor of the lens.

As a matter of fact this strategy is already at work in the design and production of modern progressive addition lenses. What as yet is actually done in a heuristic manner, can now be explained by the generalized Minkwitz theorem - at least in the direct surroundings of the principal line.

In a way the Minkwitz theorem, as seen in the perspective of the compatibility equations, is a kind of no-go-theorem: it is not possible to assign arbitrary values of optical power along a line without introducing astigmatism, either along or lateral to that line. Therefore, the canonical layout of a progressive addition lens can be tweaked, but it can not be shaken up.

REFERENCES

1. G. Esser, W. Becken, H. Altheimer, W. Müller, Generalization of the Minkwitz Theorem to Non-Umbilical Lines, JOSA A **34**, 441–448 (2017).