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RESEARCH NOTE

Oblique Central Refraction in Tilted Spherocylindrical Lenses

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ABSTRACT: Spectacle corrections worn in a frame with a faceform tilt should be used with effective spherocylindrical parameters. For the case of oblique central refraction, Keating presented a procedure to determine these parameters in a third-order approximation. The central equation of his approach, the effective dioptric power matrix, is partly the result of an educated guess. Based on the equations of wavefront tracing, an analytical derivation of his equation, slightly modified however, is given in this paper. (Optom Vis Sci 2002;79:68–73)

Key Words: spherocylindrical lenses, faceform tilt, oblique central refraction, matrix optics, wavefront tracing

Faceform tilted spectacle frames, especially sunglasses and sports goggles, are increasingly used. If they are worn with correcting spectacle lenses, the lens parameters are different from those determined in a standard refraction. Keating[1] described a procedure to calculate the effective parameters of the spectacle lenses needed to offset the optical effects introduced by the faceform tilt. In a second paper,[2] he analyzed the advanced problem of a combination of faceform and pantoscopic tilts. In both papers, Keating applied the third-order approximation for thin lenses, assuming a situation where a ray passes through the optical center of the tilted spectacle lens. This geometrical setup is well known as oblique central refraction (OCR). Exact ray tracing results, also given by Keating, back up his approach. In various applications around the world, the Keating equation for the effective dioptric power matrix has been found useful and has helped to improve the vision of many people.

However, the Keating equation (Equation 15 of the first cited paper) is partly the result of an educated guess and lacks an analytical derivation. Because Keating checked the usefulness of his equation by the numerical results from exact ray tracing, the missing foundation is not a problem, at least from a pragmatic point of view. Nevertheless, the purpose of this paper is to present a derivation of the Keating equation. As the result shows, the original Keating equation has to be modified slightly and will look a little simpler. This might help to make Keating's approach more comprehensive and plausible.

This paper is organized as follows. First, the equations of the so-called wavefront tracing are introduced. They describe the path of an infinitesimal wavefront area through an optical system. Unfortunately, this takes some space, which is needed to explain these less-known equations. Next, they are applied to the case of paraxial refraction and connected to the concept of the dioptric power matrix. The problem of OCR is solved by these equations, eventually resulting in an effective dioptric power matrix, which vindicates the result of Keating. The paper concludes with some further remarks on the problem of dispensing tilted frames. The hasty reader might directly jump to Equation 32 to grasp the main result.

WAVEFRONT TRACING EQUATIONS

In visual optics, the terminology of wavefronts and their vergences is commonplace. A neat terminology, the dioptric matrix approach, has been developed to describe optical problems concerned with spherocylindrical elements.[3, 4] There are at least two reasons for this development. First, astigmatic pencils belong to the everyday business of optometry, which is different from other branches of optics. Second, due to the relatively small pupil of the eye, we are dealing with narrow pencils, and, therefore, the local curvatures of small wavefront elements are sufficient to describe the relevant optics (i.e., parameters such as $S/C \times \alpha$) as long as higher aberrations are excluded. The appropriate tool of wavefront tracing is described in detail by Stavroudis,[5, 6] to which we refer heavily. Stavroudis uses the term generalized ray tracing. Wavefront tracing is preferred in this paper because the procedure provides information on the shape of the wavefront in a small neighborhood of its intersection with the related ray. For spherical surfaces, the Coddington equations create the equivalent information.[7] They fail, however, for spherocylindrical surfaces with oblique axes. A quite recent but more formal extension of Coddington equations to wavefront tracing is given by Landgrave and Moya-Cessa.[8]

As a first step, wavefront tracing is applied to the refraction of an (infinitesimal) wavefront Σ at a surface S, which separates media with refractive indices n and n'. The unit normal vector to the wavefront Σ, the ray direction vector, is denoted $\mathbf{s}$ and is normalized to $\mathbf{s}^2 = 1$. The normal vectors to the refracted wavefront Σ' and to the refracting surface S are called $\mathbf{s}'$ and $\mathbf{m}$, respectively. The latter depends on the point of incidence P. For all vectors, which are *all unit vectors*, small bold letters are used. For the matrices, we switch to capital bold letters. The directions of the ray direction vectors are governed by Snell's law in its vector form

$$n'\mathbf{s}' = n\mathbf{s} + g\mathbf{m} \tag{1}$$

where g is a function of the angles of incidence I and refraction I'

$$g = n' \cos I' - n \cos I \tag{2}$$

The cosines of the angles are given by

$$\cos I = \mathbf{s} \cdot \mathbf{m} \quad \text{and} \quad \cos I' = \mathbf{s}' \cdot \mathbf{m} \tag{3}$$

We next introduce three coordinate systems associated with the wavefront, the refracting surface, and the refracted wavefront. The primed quantities belong to the refracted wavefront Σ', the unprimed to Σ, and the quantities with a bar are associated with the refracting surface S. Common to all three is the vector $\mathbf{p}$, normal to the plane of incidence, tangent to both wavefronts and to the refracting surface, given by the vector product

$$\mathbf{p} = \frac{\mathbf{m} \times \mathbf{s}}{\sin I} \tag{4}$$

Next, the ray vectors $\mathbf{s}$ and $\mathbf{s}'$ and the normal vector to the refractive surface $\bar{\mathbf{m}}$ are chosen. To complete the definition, we use $\mathbf{q} = \mathbf{p} \times \mathbf{s}$, $\mathbf{q}' = \mathbf{p} \times \mathbf{s}'$ and $\bar{\mathbf{q}} = \mathbf{p} \times \bar{\mathbf{m}}$.

The pair of *principal* curvatures of the incident wavefront Σ, of the refracted wavefront Σ', and of the surface S are labeled κ_1 and κ_2, κ'_1 and κ'_2, and c_1 and c_2, respectively. Greek letters are used for wavefront curvatures, and Latin letters are used for surface curvatures. Finally, the principal directions of κ_1, κ'_1, and c_1 are given by the vectors $\mathbf{t}$, $\mathbf{t}'$, and $\bar{\mathbf{t}}$, respectively. Next we calculate the angles β, β', and $\bar{\beta}$ between the vectors $\mathbf{t}$, $\mathbf{t}'$, and $\bar{\mathbf{t}}$ and $\mathbf{p}$ according to

$$\cos \beta = \mathbf{t} \cdot \mathbf{p} \quad \cos \bar{\beta} = \bar{\mathbf{t}} \cdot \mathbf{p} \quad \cos \beta' = \mathbf{t}' \cdot \mathbf{p} \tag{5}$$

With respect to the defined coordinate systems, we are able to calculate a pair of *normal* curvatures in arbitrary orthogonal planes containing the normal vector and the related torsion for the incident wavefront by

$$\begin{aligned} \kappa_p &= \kappa_1 \cos^2\beta + \kappa_2 \sin^2\beta \\ \kappa_q &= \kappa_1 \sin^2\beta + \kappa_2 \cos^2\beta \\ \kappa_{pq} &= (\kappa_1 - \kappa_2) \cos \beta \sin \beta \end{aligned} \tag{6}$$

for the refracting surface by

$$\begin{aligned} c_{\bar{p}} &= c_1 \cos^2\bar{\beta} + c_2 \sin^2\bar{\beta} \\ c_{\bar{q}} &= c_1 \sin^2\bar{\beta} + c_2 \cos^2\bar{\beta} \end{aligned} \tag{7}$$

$$c_{\bar{p}\bar{q}} = (c_1 - c_2)\cos \bar{\beta} \sin \bar{\beta}$$

and eventually for the refracted wavefront by

$$\begin{aligned} \kappa'_{p'} &= \kappa'_1 \cos^2\beta' + \kappa'_2 \sin^2\beta' \\ \kappa'_q &= \kappa'_1 \sin^2\beta' + \kappa'_2 \cos^2\beta' \\ \kappa'_{p'q'} &= (\kappa'_1 - \kappa'_2)\cos \beta' \sin \beta' \end{aligned} \tag{8}$$

This concludes the topic of coordinate systems. The set of wavefront tracing equations can now be presented

$$\begin{aligned} n'\kappa'_p &= n\kappa_p + gc_p \\ n'\kappa'_{q'} &= \frac{\cos^2 I}{\cos^2 I'} n\kappa_q + \frac{g}{\cos^2 I'} c_{\bar{q}} \\ n'\kappa'_{pq'} &= \frac{\cos I}{\cos I'} n\kappa_{pq} + \frac{g}{\cos I'} c_{p\bar{q}} \end{aligned} \tag{9}$$

The somewhat cumbersome notation of the general case should remind the reader that all curvatures and torsions are referred to their related coordinate systems. However, the application to the problem of OCR will make things easier.

PARAXIAL REFRACTION OF A WAVEFRONT

Next, the wavefront tracing equations are related to the notion of the dioptric power matrix. Therefore, we consider the simple case of a paraxial refraction at a single spherocylindrical surface, when $\cos I = \cos I' = 1$ or $I = I' = 0$, which results in $g = n' - n$. The incident and the refracted wavefront propagate along the optical axis, called z axis, and therefore the ray vectors are equal to the surface normal

$$\mathbf{z} = \mathbf{s} = \mathbf{s}' = \mathbf{m} \tag{10}$$

All three coordinate systems, defined above, become identical. The direction of $\mathbf{p}$ is arbitrarily defined as the vertical y axis, i.e.,

$$\mathbf{y} = \mathbf{p} \tag{11}$$

Eventually, by

$$\mathbf{x} = \mathbf{q} = \mathbf{q}' = \bar{\mathbf{q}} \tag{12}$$

we end up with a right-handed coordinate system, spanned by $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$. It is worthwhile to note that the direction of the horizontal x axis is opposite to the usual coordinate system (Fig. 1). Looking from upstream in the positive z direction toward the eye, the x axis is pointing to the left to ensure a right-handed coordinate system.

The principle curvature directions of the incident wavefront are chosen to be aligned with the x and y axes. However, the principle curvature direction of the surface, c_1, includes an angle α, $0 \leq \alpha \leq \pi$, with the negative x axis, i.e.,

$$\bar{\mathbf{t}} = -\mathbf{x} \cos \alpha + \mathbf{y} \sin \alpha \tag{13}$$

Again, looking from upstream in the positive z direction, this angle α agrees with the conventional definition used for the direc-

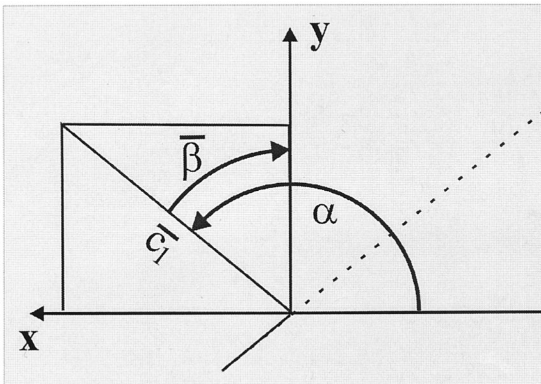

FIGURE 1.
Looking from upstream in the positive z direction toward the eye, the positive x axis points to the left to ensure a right-handed coordinate system. The angle α is measured between the direction of the cylinder axis and the *negative* x axis. The angle $\bar{\beta}$ is applied in the wavefront tracing equations.

tion of the cylinder axis. Applying these assumptions to the wavefront tracing Equations 9 result in

$$\begin{aligned} n'\kappa_x' &= n\kappa_x + (n' - n)(c_1\cos^2\alpha + c_2\sin^2\alpha) \\ n'\kappa_y' &= n\kappa_y + (n' - n)(c_1\sin^2\alpha + c_2\cos^2\alpha) \\ n'\kappa_{xy}' &= n\kappa_{xy} + (n' - n)((c_1 - c_2)\cos\alpha\sin\alpha) \end{aligned} \tag{14}$$

where the relation $\cos\bar{\beta} = \bar{\mathbf{t}} \cdot \mathbf{p} = \sin\alpha$ has been inserted into Equation 7. If we define the spherical and cylindrical powers as

$$\begin{aligned} S &= (n' - n)c_1 \\ C &= (n' - n)(c_2 - c_1) \end{aligned} \tag{15}$$

and introduce the matrix notation

$$\mathbf{S} = n\begin{pmatrix} \kappa_x & \kappa_{xy} \\ \kappa_{xy} & \kappa_y \end{pmatrix} \quad \mathbf{S}' = n'\begin{pmatrix} \kappa_x' & \kappa_{xy}' \\ \kappa_{xy}' & \kappa_y' \end{pmatrix} \tag{16}$$

then the following relation holds

$$\mathbf{S}' = \mathbf{S} + \mathbf{P} \tag{17}$$

where

$$\mathbf{P} = \begin{pmatrix} P_x & P_{xy} \\ P_{xy} & P_y \end{pmatrix} = \begin{pmatrix} S + C\sin^2\alpha & -C\cos\alpha\sin\alpha \\ -C\cos\alpha\sin\alpha & S + C\cos^2\alpha \end{pmatrix} \tag{18}$$

is the well-known dioptric power matrix.

WAVEFRONT TRACING AND OCR

Wavefront tracing will now be applied to the two surfaces of a thin spherocylindrical lens in air with both surfaces in contact. The refractive index of the material is denoted by n. The first surface, or front surface, is assumed spherical, i.e., $c_1^{(1)} = c_2^{(1)} = c^{(1)}$. The superscript $^{(1)}$ denotes the surface number. The second surface, or back surface, has the pair of principal curvatures $c_1^{(2)}$ and $c_2^{(2)}$, acting as a spherocylindrical surface. The dioptric powers along the principal directions are given by

$$\begin{aligned} S &= (n - 1)(c^{(1)} - c_1^{(2)}) \\ C &= (n - 1)(c_1^{(2)} - c_2^{(2)}) \\ S + C &= (n - 1)(c^{(1)} - c_2^{(2)}) \end{aligned} \tag{19}$$

As will be seen from the result, the order of the surfaces plays no role. Because the overwhelming majority of spherocylindrical spectacle lenses have a toric back surface, the given order is the preferred starting point.

Based at the thin lens, we consider only one right-handed coordinate system because the two coordinate systems of the front and the back surface are chosen to be identical. The optical axis is chosen as the z axis. Its positive direction points toward the eye. In the plane of the lens, we apply a horizontal and vertical coordinate system. The positive direction of the vertical y axis is upwards. Therefore, to ensure the right handedness of the (x, y, z) coordinate system, the positive direction of the horizontal x axis points to the left if we are looking downstream toward the eye.

Instead of a tilted thin lens, we equivalently consider an oblique incident ray vector **s**. The tilt angle ϕ results from the angle between the incident ray and the normal vector to the first surface, which is identical to **z**, leading to

$$\mathbf{s} \cdot \mathbf{z} = \cos\phi \tag{20}$$

The incoming ray hits the front surfaces at the origin of the coordinate system. The ray heights on both surfaces are then zero. Moreover, from the assumption of a thin lens follow the relations

$$\begin{aligned} \cos I_1 &= \cos I_2' = \cos\phi \\ \cos I_1' &= \cos I_2 \end{aligned} \tag{21}$$

where the indices refer to the number of the surfaces.

The repeated application of the wavefront tracing equations to each surface, after the customary few lines, results in

$$\begin{aligned} \kappa_{p'}' &= g(c^{(1)} - c_{\bar{p}}^{(2)}) \\ \kappa_{q'}' &= \frac{g}{\cos^2\phi}(c^{(1)} - c_{\bar{q}}^{(2)}) \\ \kappa_{p'q'}' &= -\frac{g}{\cos\phi}\, c_{\bar{p}\bar{q}}^{(2)} \end{aligned} \tag{22}$$

where

$$g = n\cos I_2 - \cos\phi \tag{23}$$

The primed quantities denote the curvatures and torsion of the wavefront leaving the thin spherocylindrical lens. Therefore, at the left side of the equations, the superscript $^{(2)}$ is suppressed. The incoming wavefront is assumed plane with vanishing curvatures and torsion.

Considering a near object instead would result in additive terms to the curvatures with the related curvatures of the incident spherical wavefront. Because these terms are independent of the tilt angle, they do not infer with the considered problem, and, therefore, we omitted this case.

Up to now, the above equations are exact, and no approximation concerning the angles has been applied. The so-called third-order approximation is now introduced by the usual expansion of the cosine terms as $\cos x \approx 1 - x^2/2$. This step renders the final result independent of the actual form, i.e., the base curve, of the lens. Rearranging terms by application of Equations 20 and Snell's law finally leads to the result

$$\begin{aligned} \kappa'_{p'} &= (n-1)(c^{(1)} - c^{(2)}_{\bar{p}})h(\phi) \\ \kappa'_{q'} &= (n-1)(c^{(1)} - c^{(2)}_{\bar{q}})\frac{h(\phi)}{\cos^2\phi} \\ \kappa'_{p'q'} &= -(n-1)c^{(2)}_{\bar{p}\bar{q}}\frac{h(\phi)}{\cos\phi} \end{aligned} \tag{24}$$

where the function

$$h(\phi) = 1 + \frac{\sin^2\phi}{2n} \tag{25}$$

has been introduced. The terms of fourth and higher orders are neglected. Nevertheless, we use the equals sign in the widespread sloppy habit. The usual convention is adopted to re-express quadratic terms in ϕ as sine-squared functions and leave the cos-terms in the denominators unexpanded. This is no claim for improved accuracy because the results are still restricted to the third-order approximation.

The normal curvatures along the x and y axes have to be related to the principle curvatures of the toric surface by Equation 7. The direction of the cylinder axis of the toric surface is given by the vector $\bar{\mathbf{t}}$ and specified in Equation 13. According to Equations 5 and 4, we have

$$\cos\bar{\beta} = \frac{1}{\sin I}\bar{\mathbf{t}}\cdot(\bar{\mathbf{m}} \times \mathbf{s}) \tag{26}$$

Using $\mathbf{s} = s_x\mathbf{x} + s_y\mathbf{y} + s_z\mathbf{z}$, where $s_x^2 + s_y^2 + s_z^2 = 1$, and applying $\bar{\mathbf{m}} = \mathbf{z}$ results in

$$\cos\bar{\beta} = \frac{1}{\sin I}(s_y\cos\alpha + s_x\sin\alpha) = \cos(\alpha - \gamma) \tag{27}$$

where $\tan\gamma = s_x/s_y$.

In the case of a faceform tilt, the tangential plane is the x, z plane and, therefore, $s_y = 0$ (Fig. 2). Because $s_x = \sin I$, this leads to

$$\cos\bar{\beta} = \sin\alpha \quad \text{or} \quad \bar{\beta} = \frac{\pi}{2} - \alpha \tag{28}$$

Inserting

$$\bar{\beta} = \frac{\pi}{2} - \alpha \tag{29}$$

into Equation 7 yields

$$\begin{aligned} c^{(2)}_x &= c^{(2)}_1 + (c^{(2)}_2 - c^{(2)}_1)\cos^2\alpha \\ c^{(2)}_y &= c^{(2)}_1 + (c^{(2)}_2 - c^{(2)}_1)\sin^2\alpha \\ c^{(2)}_{xy} &= (c^{(2)}_1 - c^{(2)}_2)\cos\alpha\,\sin\alpha \end{aligned} \tag{30}$$

where the indices p, q have been changed according to the relations $\mathbf{p} = \mathbf{y}$ and $\mathbf{q} = \mathbf{x}$. Combining this result with Equations 24 and 19 leads to

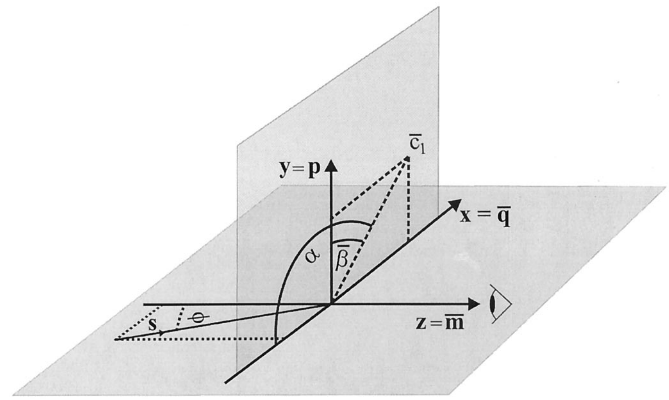

FIGURE 2.

A right-handed coordinate system is based at the thin lens. The z axis (optical axis) points toward the eye for positive values. The y axis and x axis are orientated vertically and horizontally. Shown is the incident ray vector **s** including an angle ϕ with the optical axis. The displayed situation is equivalent to a faceform tilted lens. The principle direction $\bar{c}_1$ of the lens includes an angle $\bar{\beta}$ with the y axis. The angle α, however, measured from the negative x axis is the standard angle describing the orientation of the cylinder axis.

$$\begin{aligned} \kappa'_{y'} &= (S + C\sin^2\alpha)\,h(\phi) \\ \kappa'_{x'} &= (S + C\cos^2\alpha)\,\frac{h(\phi)}{\cos^2\phi} \\ \kappa'_{x'y'} &= -C\,\frac{h(\phi)}{\cos\phi} \end{aligned} \tag{31}$$

Finally, the above equations can be identified with the effective dioptric power matrix $\mathbf{P}(\phi)$, depending on the tilt angle ϕ, resulting in the central equation of this paper

$$\mathbf{P}(\phi) = h(\phi)\begin{pmatrix} \dfrac{P_x}{\cos^2\phi} & \dfrac{P_{xy}}{\cos\phi} \\ \dfrac{P_{xy}}{\cos\phi} & P_y \end{pmatrix} \tag{32}$$

It is worth mentioning that $\mathbf{P}(\phi)$ does not depend on the sign of ϕ because $\mathbf{P}(\phi) = \mathbf{P}(-\phi)$ is an even function of the tilt angle. Additionally, $\mathbf{P}(\phi)$ reduces to $\mathbf{P}_0$ in the case of no tilt, $\phi = 0$. Our main result may also be written more formally as

$$\mathbf{P}(\phi) = \mathbf{T}(\phi)\,\mathbf{P}_0\,\mathbf{T}(\phi) \tag{33}$$

where

$$\mathbf{T} = \sqrt{h(\phi)}\begin{pmatrix} \dfrac{1}{\cos\phi} & 0 \\ 0 & 1 \end{pmatrix} \tag{34}$$

In case of a pantoscopic tilt, with $s_x = 0$, which is not considered further here, the diagonal elements of $\mathbf{T}$ have to be exchanged according to the exchanged directions of **p** and **q**. However, the case of a spectacle lenses with a pantoscopic tilt only is of little interest. If the lens is mounted according to the center of rotation condition,[9] i.e., the optical axis of the lens passes through the eye's center of rotation, the angle of incidence at the optical center of the lens is zero. Simply, there is no *central* refraction to be dealt with.

The mixed case of pantoscopic and faceform tilts needs more attention. A thorough analysis of this situation, however, is given by Keating.[2]

COMPARISON WITH THE KEATING EQUATION

Keating's result for the effective dioptric power matrix, Equation 15 in his paper,[1] is given by

$$\mathbf{P}(\phi) = \begin{pmatrix} P_x T_c & P_{xy} H_c \\ P_{xy} H_c & P_y S_c \end{pmatrix} \tag{35}$$

He used the abbreviations

$$S_c = 1 + \frac{\sin^2\phi}{2n} = h(\phi) \tag{36}$$

$$T_c = \frac{2n + \sin^2\phi}{2n\cos^2\phi} = h(\phi)/\cos^2\phi \tag{37}$$

The diagonal elements of his matrix are therefore identical with those of Equation 32. The off-diagonal elements are slightly different, because Keating assumed in Equation 16 of his paper the arithmetic mean

$$H_c = \tfrac{1}{2}(S_c + T_c) = h(\phi)\left[\frac{1}{2}\left(1 + \frac{1}{\cos^2\phi}\right)\right] \tag{38}$$

to multiply the elements P_{xy}. However, our result, Equation 32, shows that the geometric mean instead leads to the correct factor

$$\sqrt{S_c T_c} = \sqrt{h^2(\phi)/\cos^2\phi} = h(\phi)/\cos\phi \tag{39}$$

Therefore, the Keating equation has to be slightly modified by substituting the factor 38 by the factor 39. The numerical difference between both factors makes up to 0.012 at an angle of $\phi = 30°$. Matrix elements of the order of 20 D are necessary to reach the clinical threshold of 0.25 D. Therefore, the numerical difference is clearly negligible for all practical purposes. With higher angles, this difference will increase. However, the whole approach will then loose accuracy due to the limitation of the third-order approximation. Therefore, the wave tracing equations lead to a vindication of the Keating equation for the effective spherocylindrical parameters of a tilted lens.

COMPENSATING PARAMETERS

In this section, we repeat the procedure for determining the parameters of a compensating lens using the slightly modified Keating equation. If the tilted spectacle lens must match an R_x of $S/C \times \alpha$ with a related dioptric power matrix

$$\mathbf{P}_{R_x} = \begin{pmatrix} P_x & P_{xy} \\ P_{xy} & P_y \end{pmatrix} \tag{40}$$

then the effective power matrix of the tilted lens $\mathbf{P}(\phi)$ must be equal to $\mathbf{P}_{Rx}$ to reproduce the desired prescription values in the tilted position. Therefore, the spherocylindrical parameters of a compensating lens must be determined from the matrix

$$\mathbf{P}^{(c)} = \begin{pmatrix} P_x^{(c)} & P_{xy}^{(c)} \\ P_{xy}^{(c)} & P_y^{(c)} \end{pmatrix} = \mathbf{T}^{-1}(\phi)\mathbf{P}_{R_x}\mathbf{T}^{-1}(\phi) \tag{41}$$

or explicitly

$$\mathbf{P}^{(c)} = \frac{1}{h(\phi)}\begin{pmatrix} P_x\cos^2\phi & P_{xy}\cos\phi \\ P_{xy}\cos\phi & P_y \end{pmatrix} \tag{42}$$

where P_x, P_y, and P_{xy} belong to the given R_x values. The compensating prescription values are found from the relations

$$C^{(c)} = \pm\sqrt{[\mathrm{Tr}(\mathbf{P}^{(c)})]^2 - 4\det(\mathbf{P}^{(c)})}$$
$$S^{(c)} = \tfrac{1}{2}(\mathrm{Tr}(\mathbf{P}^{(c)}) - C^{(c)}) \tag{43}$$
$$\tan\alpha^{(c)} = \frac{S^{(c)} - P_x^{(c)}}{P_{xy}^{(c)}}$$

where the trace of a matrix is defined as

$$\mathrm{Tr}(\mathbf{P}^{(c)}) = P_x^{(c)} + P_y^{(c)} \tag{44}$$

and the determinant is given by

$$\det(\mathbf{P}^{(c)}) = P_x^{(c)}P_y^{(c)} - P_{xy}^{(c)}P_{xy}^{(c)} \tag{45}$$

This nontrivial calculation of the parameters $S^{(c)}/C^{(c)} \times \alpha^{(c)}$ makes it difficult to imagine how this compensation could be determined without the help of the Keating equation, or more generally, without the help of the dioptric power matrix approach.

EFFECTIVE TILT ANGLE AND PRISMATIC EFFECTS

In the above considerations, it is assumed that the tilt angle is known. However, its determination might be a more complicated story. As a first step, the tilt angle of the faceform tilted frame may be used as a proxy to the tilt angle of the lens itself. The frame-related angle is easily determined by using a scanner or a copy machine to image a projection of the frame. This image allows the tilt angle of the frame to be evaluated, either by hand or using software, to an accuracy of around 1°.

Next, the lenses have to be fitted to the frame according to the interpupillary distance. Remember that the fitting coordinates change due to the projective geometry of the tilted frame. If the lens has to be cut unsymmetrically, which is nearly always the case, and a facette parallel to the front surface is applied, then this might lead to an effective rotation of the lens, resulting in a difference between the frame tilt and the lens tilt (J. Hoffmann, personal communication). Moreover, this difference also depends on the base curve F_1 of the lens and on the lens center thickness d. Without further information on the geometry of the lens, the problem of the effective tilt angle cannot be solved. However, a thorough investigation of these problems is beyond the scope of this article.

As soon as the lens is not thin in the ideal sense, the tilted lens will introduce prismatic effects. Proceeding against the light direction with a ray height of zero at the back surface, the paraxial approximation estimates the prismatic effect p as

$$p = 100\,\frac{d}{n}\,F_1\phi^{\Delta} \tag{46}$$

where the tilt angle ϕ is measured in radians. For example, a base curve of 8 D, a reduced thickness of d/n = 3 mm, and a tilt angle of $\phi = 0.35$ (20°) induces a prismatic effect of 0.8Δ for each eye, adding up to a horizontal prismatic load of 1.6Δ base-out, which should be offset by a compensating prismatic value of opposite orientation, namely base-in. The direction of the prismatic load is independent of the dioptric power. Even tilted frames worn without any correction suffer from this effect. Furthermore, the sign of the correction has no influence on the direction of the resulting prismatic load. Again, data from the actual lens are needed to provide a more accurate estimate.

CONCLUSION

The Keating equation enables the correct dispensing of spectacle lenses within faceform tilted frames. An analytical derivation of this equation has been given in this paper, leading to a slight modification of the Keating equation, which can be neglected from a numerical point of view.

A word of caution concludes this paper. Beside the prismatic effects, which have to be dealt with separately, the application of the Keating equation strictly compensates the effects of a tilt under the assumption of the OCR for one point only. The field of view of optimal vision is restricted by aberrations that rise quickly if the line of sight lies away from the optical center of the lens. Therefore, in cases of tilt angles as large as 30° and more, which are favored by the most fashionable frames, they should not be dispensed with tilted *correcting* glasses if they are worn in situations where undisturbed visual information from a larger field of view is critical.

APPENDIX

Immersed Tilted Lens

The above results are limited to tilted lenses in air. It might be interesting to consider an immersed tilted lens, e.g., an intraocular lens, which may also suffer tilting *in situ*. In this case, where the lens, with index n, is surrounded by a medium of index n*, which is the same on both sides of the lens, the above results can be generalized in a straightforward way. Equations 19 and 25 have to be substituted by the following expressions

$$S = (n - n^*)(c^{(1)} - c_1^{(2)}) \tag{47}$$

$$C = (n - n^*)(c_1^{(2)} - c_2^{(2)})$$

$$S + C = (n - n^*)(c^{(1)} - c_2^{(2)})$$

$$h(\phi) = 1 + \frac{1}{2}\frac{n^*}{n}\sin^2\phi \tag{48}$$

The case in which the index is not the same on either side of the lens, as in the case of the cornea for example, is more complicated. This situation needs a detailed study and is not considered further in this paper.

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