1040-5488/04/8111-0880/0 VOL. 81, NO. 11, PP. 880–883
OPTOMETRY AND VISION SCIENCE

ORIGINAL ARTICLE

Tolerating Vertex Distance Changes for Spherocylindrical Corrections

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ABSTRACT: Prescriptions depend on the vertex distance. Although weak prescriptions are insensitive to a vertex distance change, stronger ones must be adjusted if the vertex distance is modified by an appreciable amount. The tolerable amount can be specified for pure spherical powers. For spherocylindrical corrections, the concept of dioptric distance is invoked in this article and applied to the propagation of an astigmatic wavefront. This leads to a simple rule that is well suited to decide on the necessity of modifying a prescription. (Optom Vis Sci 2004;81:880–883)

Key Words: prescription, vertex distance, dioptric distance, spherocylindrical lenses, propagation of astigmatic wavefronts

Spectacle prescriptions often report only three quantities: sphere (S), cylinder (C), and axis (α) in the form $S + C \times \alpha$. Prismatic corrections may call for additional numbers that are not of interest here. Depending on the strength of the prescription, the spatial position in which the prescription has been determined [i.e., the vertex distance (d)] should be mentioned as well because the propagation of the wavefront along a larger or smaller vertex distance changes the effective applied correction.

As an extreme but common example, consider a contact lens that substitutes for a spectacle lens. In this case, the vertex distance changes by roughly 15 mm, and sometimes the prescription has to be recalculated. For high-powered lenses, even the difference between results using a phoropter (vertex distance, typically $d \approx 20$ mm) or a trial frame ($d \approx 12$ mm) may be important.

In daily practice, a common rule of thumb is applied to check for the necessity of adjusting prescriptions. Given a pure spherical power with $|S| < 3.00$ D, the vertex distance is not an important issue. Therefore, spectacle and contact lenses are exchangeably applying the same correction. In reverse, for $|S| \geq 3.00$ D, the prescription should be modified according to the applied vertex distance. This rule of thumb can be derived, as shown below, from two assumptions. First, a defocus of 0.125 D is admitted. Second, the vertex distance changes by no more than 15 mm.

However, astigmatic corrections are more complicated. A logical approach is to transfer the rule of thumb to each principal meridian. To do this, the rule has to be reformulated. As long as the absolute power of each principal meridian is <3.00 D, a vertex distance change up to 15 mm can be tolerated. According to this rule, the region of acceptable spherocylindrical combinations is described by a square in a coordinate system representing the power in each meridian. The region of acceptable spherocylindrical combinations transforms into a triangle when considering only plus cylinder values, as shown in Fig. 1.

This principal meridian approach, as it may be termed, has the advantage of simplicity. However, refractive errors with different impacts on the visual acuity are treated as being equivalent. For example, consider the following two combinations of magnitudes for the principal meridians: (1) 2.75 D/2.50 D and (2) 3.00

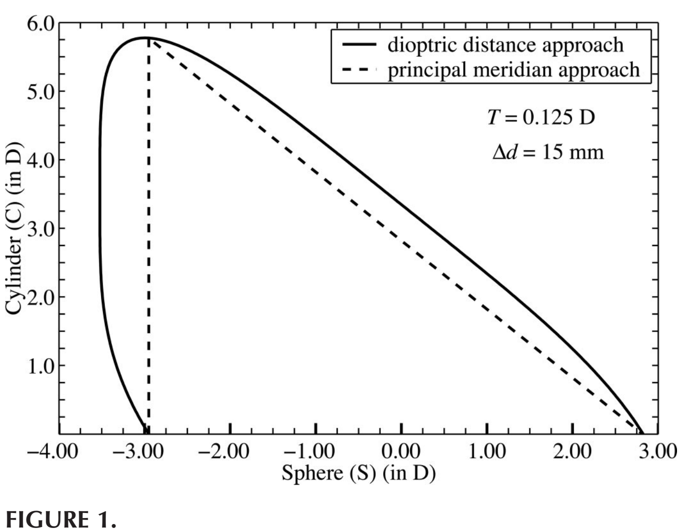

FIGURE 1.

The comparison of the two approaches (see text) shows curves that share a coarse similarity. The numerical results are different: The acceptable prescriptions can be off by a maximum of 0.5 DS or 0.5 DC. A vertex distance change of $\Delta d = 15$ mm is assumed, and a tolerance of $T = 0.125$ D is applied.

D/0.00 D, with arbitrary axis orientation. If a wavefront with those curvatures travels downstream toward the eye, both curvatures increase for the first example (1). Accordingly, the spherical equivalent power will increase as well. However, the cylinder power will change little. Therefore, compared with the starting wavefront, the difference is mainly spherical. Now change from a nearly spherical wavefront to the second example (2) with a significant cylinder power. While traveling downstream, the positive curvature of the wavefront will increase for one meridian but will remain unaltered for the other (0 D). In other words, the spherical equivalent power will increase little, roughly one-half the value of the preceding example, although the cylinder power will be clearly enhanced. Therefore, now the difference is mainly caused by a changed cylinder power. According to the principal meridian approach, the first example (1) can be tolerated, whereas the second example (2) cannot. However, the impact on visual acuity in both cases is clearly different. If this argument is taken into consideration, both examples can be tolerated, as will be shown below.

Generally speaking, a vertex distance change leads to a different effective spherocylindrical combination that has to be compared with the prescribed one. The distance between these two spherocylindrical combinations (or corresponding wavefronts) indicates a refractive error. According to this mismatch, a decision can be made whether the related vertex distance change can be tolerated. Because two spherocylindrical combinations cannot be simply subtracted to get the difference, a measure describing this difference and the impact of a given spherocylindrical refractive error on visual acuity is needed.

There are many possibilities to relate wavefront errors to their impact on visual acuity. This problem is under ongoing research.[1] I will not include a detailed discussion of this topic but prefer a pragmatic way that satisfies the following two conditions. It must be simple enough to allow for simple mathematical equations, and it can be related to visual acuity data in a meaningful way. At hand is the concept of dioptric strength introduced by Harris,[2] which is close to the power (or blur) vector concept of Thibos.[3] Both approaches yield essentially similar formulas and allow for a sufficient empirical description of the relation between refractive errors and visual acuity, as has been shown by Raasch.[4] In our case, both approaches have to be tailored to the case of propagating wavefronts. The power vector approach calls for complicated equations in this case.[5] Therefore, the dioptric strength concept is preferred and will be slightly generalized to the notion of a dioptric distance between two wavefronts. Eventually, we will arrive at a simple rule that is easily applied.

Finally, it should be emphasized that the considered problem is not how to modify corrections according to a vertex distance change. This is a well-known procedure although rarely applied in practice. The question is when this procedure can be safely neglected.

METHODS

Dioptric Distance between Two Wavefronts

We now introduce the concept of the dioptric distance between two wavefronts, which are described by vergence matrices. As an example, we consider the case of a prescription valid for an actually worn spectacle lens with a given vertex distance d. A wavefront leaving the lens is described by the back vertex vergence. Note that no assumption is made whether this wavefront results from a distant or near object. However, only for a distant object the back vertex vergence is identical to the back vertex power. In case of an astigmatic wavefront, we have to use the following symmetric back vertex vergence matrix[6,7]

$$\mathbf{V} = \begin{pmatrix} S + C\sin^2\alpha & -C\cos\alpha\sin\alpha \\ -C\cos\alpha\sin\alpha & S + C\cos^2\alpha \end{pmatrix} = \begin{pmatrix} V_{11} & V_{12} \\ V_{12} & V_{22} \end{pmatrix} \tag{1}$$

The values of S, C, and α refer to the plane of the spectacle lenses. The vertex distance d is given by the distance from the back vertex of the lens to the cornea. Now a contact lens with a different vertex distance $d^* \approx 0$ should do an equivalent job regarding the wavefront curvatures. The change in the astigmatic wavefront while traveling the distance $\Delta d = d - d^* = d$ has to be corrected for, and a new vergence matrix $\mathbf{V}^*(\Delta d)$ has to be applied to calculate the new prescription. All the quantities related to the new vertex distance are marked with an asterisk.

To determine the new vergence matrix $\mathbf{V}^*$, we start with the diagonal vergence matrix $\mathbf{V}$, according to the vertex distance d. Consider a coordinate system aligned with the principal meridians. This leads to the following diagonal representation

$$\mathbf{V} = \begin{pmatrix} S & 0 \\ 0 & S + C \end{pmatrix} \tag{2}$$

The effects of the downstream propagation of the wavefront can be reckoned separately in the two principal meridians according to

$$S^*(\Delta d) = \frac{S}{1 - \Delta d\, S} \tag{3}$$

$$(S + C)^*(\Delta d) = \frac{S + C}{1 - \Delta d\,(S + C)} \tag{4}$$

In matrix notation, this reads

$$\mathbf{V}^*(\Delta d) = \begin{pmatrix} \dfrac{S}{1 - \Delta d\, S} & 0 \\ 0 & \dfrac{S + C}{1 - \Delta d\,(S + C)} \end{pmatrix} \tag{5}$$

It is assumed that the orientation of the principal meridians do not change while the wavefront is traveling. This assumption has been proven true by several authors[8,11] and may be made plausible by inspecting the general result[12] (Eva Acosta, personal communication, November 2003) for an arbitrary coordinate system given by

$$\mathbf{V}^*(\Delta d) = \frac{1}{\gamma}(\mathbf{V} - (\Delta d)(\det \mathbf{V})\mathbf{I}) \tag{6}$$

where $\mathbf{I}$ is the unity matrix and

$$\begin{aligned} \gamma &= 1 - (\Delta d)(V_{11} + V_{22}) + (\Delta d)^2(V_{11}V_{22} - V_{12}^2) \\ &= 1 - (\Delta d)\, tr\mathbf{V} + (\Delta d)^2(\det \mathbf{V}) \end{aligned} \tag{7}$$

To decide whether a prescription has to be modified, we need a scalar quantity to represent the difference between the two wavefronts. Let us consider the concept of dioptric strength,[2] which is applied to the difference of two vergence matrices. In this case, it may be preferable to speak of a dioptric distance between two wavefronts. The dioptric distance ΔA is defined by the Frobenius norm of the difference of the related vergence matrices

$$\Delta A(\Delta d) = \frac{1}{\sqrt{2}}\|\mathbf{V}^* - \mathbf{V}\|$$

$$= \frac{1}{\sqrt{2}}\sqrt{(V^*_{11} - V_{11})^2 + 2(V^*_{21} - V_{21})^2 + (V^*_{22} - V_{22})^2} \quad (8)$$

We normalize by

$$1/\sqrt{2} \quad (9)$$

because then a pure spherical distance of 1 D leads to $\Delta A = 1$ D as well. Harris[9] has suggested the same normalization.

Now we are able to calculate the dioptric distance. We again apply a coordinate system aligned with the principal meridians. Then both matrices are diagonal ($V_{12}=V^*_{12} = 0$), and by inserting equations 2 and 5 into the definition 8, we arrive at the general solution

$$\Delta A(\Delta d) = \frac{1}{\sqrt{2}}\sqrt{\left(\frac{S}{1-(\Delta d)S} - S\right)^2 + \left(\frac{S+C}{1-(\Delta d)(S+C)} - (S+C)\right)^2} \quad (10)$$

which obviously does not depend on the orientation of the principal meridians. It is worth emphasizing this fact because the axis orientation can be excluded from all the following considerations. The derived result describes the dioptric distance between two vergence matrices related to the start point and endpoint of a propagating astigmatic wavefront, which traveled downstream by a distance Δd. In case of an upstream propagation, $\Delta d < 0$ has to be applied.

Numerically it is not difficult to evaluate the above expression. However, the general result is a bit awkward. Because in practice Δd takes small numerical values, we linearize equation 10 in Δd to approach to a simpler solution. After a Taylor expansion and some algebra, the following nice and simple expression evolves

$$\Delta A(\Delta d) \approx \frac{|\Delta d|}{\sqrt{2}}\sqrt{S^4 + (S+C)^4} \quad (11)$$

The approximated result is still invariant under a transposition from plus to minus cylinders, as is required for a meaningful ophthalmic property.[10] Furthermore, it does not depend on the sign of Δd or on the sign of the powers in the principal meridians. This is not true for the exact solution. Numerically the difference between the exact and the approximated result is <0.035 D for all the cases discussed in the following section.

RESULTS AND CONCLUSION

Assuming a tolerance T for the dioptric distance between the two vergence matrices, we establish the following condition

$$A(\Delta d) \le T \quad (12)$$

which should hold for all the corrections in which a modification is not necessary. A reasonable numerical value is chosen by

$$T = 0.125 \text{ D} \quad (13)$$

which is one-half of the smallest spacing between two spherical lenses in the typical trial case. This value is considered as insignificant in clinical practice.

First, we apply this tolerance to a spherical correction and assume a typical value of $\Delta d = 15$ mm for the vertex distance change. Then from equations 11, 12, and 13, the spherical power should obey the relation

$$|S| \le \sqrt{\frac{T}{|\Delta d|}} = 2.89 \text{ D} \quad (14)$$

Rounded to the usual 0.25 D, we get the aforementioned rule of thumb: ISI $<$ 3.00 D.

In the second step, the tolerance T is applied to an arbitrary combination of S and C. Note that the tolerance equation 12 does not require any modifications in case of spherocylindrical corrections. The chosen metric, equation 8, includes these cases automatically. We apply the same numerical value for the tolerance, $T = 0.125$ D. Note that this means neither 0.125 DS nor 0.125 DC but a dioptric distance that may be caused by a mixture of spherical and cylindrical components. Given the tolerance, we have to evaluate equation 10. The numerical results are visualized in Fig. 2 for different values of vertex changes Δd.

As an example, consider a change from spectacle lenses to contact lenses and therefore a related vertex change of $\Delta d = 15$ mm. Every combination of S and C enclosed by the solid contour line can be accepted without changing the prescription. Smaller values of Δd lead to a larger acceptable region of corrections. Vice versa, larger values of Δd restrict the tolerable area and therefore tolerable combinations of S and C. Fig. 2 demonstrates the necessity to modify prescriptions when the vertex distance is changed.

The invariance under spherocylindrical transposition causes all the curves to be point symmetric to the origin. Furthermore, the slope of all the curves at zero cylinder is fixed to a numerical value of -2 by the same invariance condition,[10] which can be nicely seen in Fig. 2.

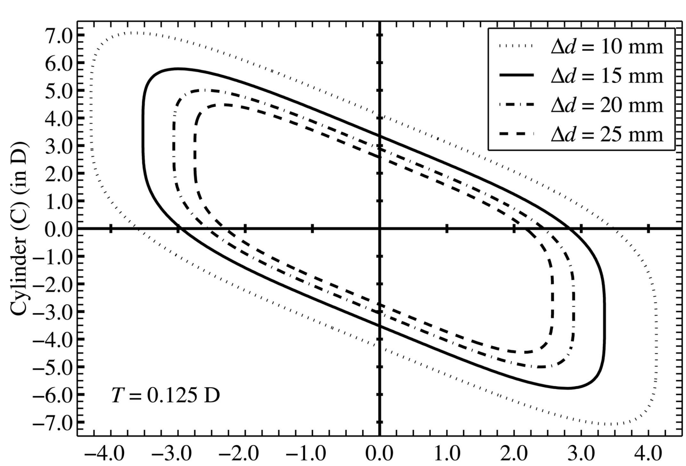

FIGURE 2.

The regions of spherocylindrical combinations that are tolerable for a given vertex distance change Δd are located inside the related curves. The dioptric distance is tolerated by $T = 0.125$ D. A change from spectacle to contact lenses is typically represented by the solid line ($\Delta d = 15$ mm). As an example, there is no need to modify a correction of $S = -3$ D and $C = +5$ D if a change of $\Delta d = 15$ mm is considered. However, for a change of $\Delta d = 20$ mm, the correction should be recalculated.

The results of the principal meridian and the dioptric distance approach are compared in Fig. 1, restricted to plus cylinders only. Beside obvious numerical differences, there exists a coarse similarity between both graphs. They look more or less like a triangle and share a long vertical portion and a nearly linear part, which starts at a maximum cylinder value and decreases to zero. Both curves share common points. These points are located at zero cylinder (spherical wavefront) and near-zero spherical equivalent power (crossed cylinder). In both cases, the resulting refractive error according to the propagation of the wavefront is nearly spherical. This is why both approaches yield similar results.

The area enclosed by the dashed curve (principal meridian approach) includes a smaller range of corrections than the area determined by the solid line (dioptric distance approach). The numerical difference in the related prescriptions can be as large as 0.5 DS spherical power or 0.5 DC cylinder power. These differences appear in a systematic way. The application of the rule of thumb (principal meridian approach) never overestimates the tolerable region of spherocylindrical combinations. This fact renders the rule of thumb applied to each meridian suitable for daily practice. The dioptric distance approach may seem more complicated, but it delivers the reason why the rule of thumb is acceptable for the practitioner.

It may be worthwhile to reconsider the question of axis orientation. In the perspective of geometrical optics, the propagation of an astigmatic wavefront clearly does not show any dependence on the orientation of the principal meridians. However, because of retinal and neural effects, the visual acuity depends on the orientation of objects. For example, the gap of a Landolt ring is less likely recognized at 45° than in a vertical or horizontal position. As long as there is no simple description of angle-dependent effects, it seems too complicated to include such phenomena into the considered problem of tolerancing.

Numerical Example

Consider the spherocylindrical combination of $S = -3.00$ DS and $C = 5.50$ DC, which is positioned in the upper left region of Fig. 2. The axis orientation is not important in the following, and for simplicity I chose $\alpha = 0$. The diagonal back vertex vergence then reads

$$\mathbf{V} = \begin{pmatrix} -3.00\ \mathrm{D} & 0.00\ \mathrm{D} \\ 0.00\ \mathrm{D} & 2.50\ \mathrm{D} \end{pmatrix} \tag{15}$$

For a vertex distance change of $\Delta d = 15$ mm, from equation 5 the following expression

$$\mathbf{V}^*(\Delta d) = \begin{pmatrix} -2.87\ \mathrm{D} & 0.00\ \mathrm{D} \\ 0.00\ \mathrm{D} & 2.60\ \mathrm{D} \end{pmatrix} \tag{16}$$

is found. The difference of both vergence matrices, divided by $\sqrt{2}$, leads to the dioptric distance

$$\Delta \mathrm{A} = \frac{1}{\sqrt{2}} \left\| \begin{pmatrix} -2.87\ \mathrm{D} & 0.00\ \mathrm{D} \\ 0.00\ \mathrm{D} & 2.60\ \mathrm{D} \end{pmatrix} - \begin{pmatrix} -3.00\ \mathrm{D} & 0.00\ \mathrm{D} \\ 0.00\ \mathrm{D} & 2.50\ \mathrm{D} \end{pmatrix} \right\| \tag{17}$$

$$= \frac{1}{\sqrt{2}} \left\| \begin{pmatrix} 0.13\ \mathrm{D} & 0.00\ \mathrm{D} \\ 0.00\ \mathrm{D} & 0.10\ \mathrm{D} \end{pmatrix} \right\| \tag{18}$$

$$= \sqrt{(0.13\ \mathrm{D})^2 + (0.10\ \mathrm{D})^2} = 0.12\ \mathrm{D} \tag{19}$$

Alternatively, by applying the approximation, equation 11, we arrive at

$$\Delta \mathrm{A}(\Delta \mathrm{d}) \approx \frac{0.015}{\sqrt{2}} \sqrt{3.00^4 + 2.50^4}\ \mathrm{D} = 0.12\ \mathrm{D} \tag{20}$$

This figure does not differ from the exact result within the displayed precision. Both values are below the threshold of $T = 0.125$ D.

ACKNOWLEDGMENTS

Supported by the ZFE at the Darmstadt University of Applied Sciences.

Received February 3, 2004; accepted August 18, 2004.

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