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CONFERENCE PAPER

Hans-Heinrich Fick: Early contributions to the theory of astigmatic systems†

Hans-Heinrich Fick (1922 - 1996) *

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Introduction

As early as 1972 the German ophthalmic optician Hans-Heinrich Fick (1922-1996) published a series of 22 papers[1] to introduce the idea of the dioptric power matrix.The series was entitled *Progressive Calculation Methods in Optometry* Fortschrittliche Rechnungsarten in der Augenoptik). In his contributions he derived concepts like the dioptric power matrix, the matrix of back vertex power and the linear optics version of the Gullstrand equation. Furthermore, HHF discussed applications like anamorphic effects, crossed cylinders and re-formulated Prentice's equation.

Already four years earlier, in September 1968, HHF communicated most of his results by a letter to the German Professor, J. Flügge, and asked him to review his approach. Flügge appreciated HHF's work and recommended publication of his findings in the German journal *Optik*. For unknown reasons HHF waited until 1972 and published his results in a more didactic rather than rigorous style in the internationally unknown journal *Der Augenoptiker.* According to my knowledge this journal was not included in any citation index.

Astonishingly HHF had no academic education but was trained as an optician and watchmaker. Driven by his optical intuition he acquired an enormous mathematical knowledge by autodidactic training. He tried hard to explain his ideas in simple ways and always emphasized the practical importance of his approach which later on was termed linear optics of astigmatic systems. Nevertheless, his ideas were not accepted in Germany at that time. Because HHF published in German and additionally in a journal more related to the craftsman than to a scientific readership his approach was nearly forgotten.[2] However, the field of linear optics has grown enormously in the last two decades, and it seems appropriate to give HHF the credit he deserves for his early work in this field.

Two caveats conclude this introduction. (1) The German optometrist (Augenoptiker) is different from the optometrist in the Anglo-Saxon world. To put it briefly the German version includes more opticianry and technical optics and less medical aspects.[3] (2) No attempt is made to link HHF's work to later published results. The reason is simple: HHF and later authors belonged to disjunct sets in time and space and they most probably did not know anything of each other.

Progressive Calculation Methods in Optometry (PCMO)

The PCMO appeared monthly in 18 succeeding contributions from January 1972 to June 1974. The series comprises a total of 70 pages and was addressed to the traditional German optometrist. Instead of mathematical rigor, intuitive arguments and examples are the preferred style. Therefore, 50% of the text is worked out problems and homework exercises with solutions published a

* Photograph of Hans-Heinrich Fick from the early 1990's, used with kind permission of Dorothea Fick.

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month later. Keep in mind that at those times numerical work was mainly done with a slide rule.

The PCMO is divided into nine chapters which are subdivided into sections. This structure dominates the unusual numbering of equations. The first two digits, separated by a colon, represent chapter and section. The third digit counts the equations in the section, while the fourth digit represents variants of an equation. For example: formula 5,324 is found in chapter 5, section 3 as the fourth variant of the second equation. HHF uses overlined letters for vectors and a letter in brackets denotes a matrix. However, in this contribution bold small letters are used for vectors and bold capital letters for matrices. In this paper a rough survey of the main results of HHF's work in the PCMO is presented. It goes without saying that in the field of linear optics the assumptions of paraxial optics are omnipresent.

As the title PCMO indicates, emphasis is laid on the calculation method. Nearly half of the series is spent on an introduction to the very basics of vector and matrix algebra in two dimensions. Most of the required tools are introduced on the fly. For example the vectorial approach is demonstrated in the field of prismatic effects. As a first example, which goes beyond the well known knowledge at that time, the combination of three obliquely crossed cylinders in direct contact is discussed in vector notation. The example is aimed at the combination of an astigmatic eye, a spectacle lens and a trial lens. HHF's approach was superior to the traditional knowledge prevailing at that time.

The key example leading by analogy to the dioptric power matrix is the magnification matrix for an anamorphic system. Probably, HHF was driven by binocular vision problems sometimes encountered with astigmatic corrections. An anamorphic system has two orthogonal principal meridians, denoted by u and v, with the principal magnifications V_u and V_v. For an object point represented by a column vector $\mathbf{r} = (x, y)^T$ and including an angle β with the direction of V_u the mapping to the image vector $\mathbf{r}'$ is described by the matrix equation $\mathbf{r}' = \mathbf{V}\,\mathbf{r}$, where

$$\mathbf{V} = \begin{pmatrix} V_{xx} & V_{xy} \\ V_{yx} & V_{yy} \end{pmatrix} \tag{1}$$

and

$$V_{xx} = V_u \cos^2\beta + V_v \sin^2\beta \tag{2}$$

$$V_{xy} = V_{yx} = (V_u - V_v) \sin\beta \cos\beta \tag{3}$$

$$V_{yy} = V_u \sin^2\beta + V_v \cos^2\beta \tag{4}$$

Replacing V_u and V_v by the principal powers of an astigmatic lens, D_u and D_v, we are lead directly to the dioptric power matrix. The coupling of two anamorphic systems is described by matrix multiplication. In this case the resulting matrix might be asymmetric, as HHF mentions. He concluded that in general a system of two anamorphic elements cannot be replaced by a single anamorphic system.

To introduce the dioptric power matrix HHF started with Prentice's rule applied in the two principal meridians. In vol. 12 /1972 on page 62, eq. 4,22, HHF ended up with a generalized form of Prentice's rule and the dioptric power matrix

$$\mathbf{p} = -\mathbf{D}\,\mathbf{r}_0 \tag{5}$$

The vector $\mathbf{r}_0$ represents the ray coordinates in the plane of the lens. HHF uses cm as unit for the ray coordinates and m^{-1} for the elements of the dioptric power matrix $\mathbf{D}$, which is given by

$$\mathbf{D} = \begin{pmatrix} D_{xx} & D_{xy} \\ D_{yx} & D_{yy} \end{pmatrix} \tag{6}$$

and

$$D_{xx} = D_u \cos^2\beta + D_v \sin^2\beta \tag{7}$$

$$D_{xy} = D_{yx} = (D_u - D_v) \sin\beta \cos\beta \tag{8}$$

$$D_{yy} = D_u \sin^2\beta + D_v \cos^2\beta \tag{9}$$

HHF applied Prentice's rule in this new form to the problems of prismatic effect in spectacle lenses.

At the moment when the sum or the product of two matrices enter the picture, the question of decoding the resulting matrix elements arises. To retrieve the traditional values of sphere, cylinder and axis (SCA-values) HHF derived the following equations (5,511. . . 5,554) for a general matrix $\mathbf{A}$:

$$A_u - A_v = \pm \sqrt{(\mathrm{tr}(\mathbf{A}))^2 - 4 \det(\mathbf{A})} \tag{10}$$

$$A_u = \tfrac{1}{2}\left[(A_{xx} + A_{yy}) + (A_u - A_v)\right] \tag{11}$$

$$A_v = \frac{1}{2}\,[(A_{xx} + A_{yy}) - (A_u - A_v)] \qquad (12)$$

$$\tan\beta_u = \frac{A_u - A_{xx}}{A_{xy}} = \frac{A_{yx}}{A_u - A_{yy}} \qquad (13)$$

$$\tan\beta_v = \frac{A_v - A_{xx}}{A_{xy}} = \frac{A_{yx}}{A_v - A_{yy}} \qquad (14)$$

The case of asymmetric matrices, where the principal directions need not be orthogonal, is explicitly included. However, only for symmetric matrices, indicated by the letter **D**, a cylinder can be defined as

$$\pm Z = D_u - D_v \qquad (15)$$

In the next step the combination of obliquely orientated toric lenses, instead of crossed cylinders, in direct contact is considered. Starting from Prentice's equation in matrix notation, eq. (5), and the condition $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2$ for two lenses in direct contact, yields

$$\mathbf{D}\mathbf{r}_0 = [\mathbf{D}_1 + \mathbf{D}_2]\,\mathbf{r}_0 \qquad (16)$$

Assuming an arbitrary $\mathbf{r}_0$ one reaches the conclusion

$$\mathbf{D} = \mathbf{D}_1 + \mathbf{D}_2 \qquad (17)$$

This equation is the generalization of the well known case of spherical lenses. The rule is simple: replace the scalar refractive power of spherical lenses by the dioptric power matrix of astigmatic lenses. This simplicity is the very importance of the introduced matrix formalism. Concerning computation time, the numerical effort is not really reduced. However, there are two advantages. Firstly, the result is easily generalized to more than two lenses in direct contact (6,21)

$$\mathbf{D} = \mathbf{D}_1 + \mathbf{D}_2 + \mathbf{D}_3 + \ldots \qquad (18)$$

Secondly, the addition rule is correct for an arbitrary sphero-cylindrical combination. There is no need for a hitherto separate treatment of the spherical and cylindrical components. By decoding the resulting matrix of two obliquely crossed cylinders, which include an angle ε, HHF re-derived the known equations

$$Z^2 = Z_1^2 + Z_2^2 + 2Z_1Z_2\cos 2\varepsilon \qquad (19)$$

$$S = \frac{1}{2}\,(Z_1 + Z_2 - Z) \qquad (20)$$

$$\tan\phi = \frac{Z - Z_1 - Z_2\cos 2\varepsilon}{Z_2\sin 2\varepsilon} \qquad (21)$$

The next topic leads us to inversematrices and the question: How much decentration is needed to induce a given prismatic effect? The answer is given by solving eq. (5) for $\mathbf{r}_0$

$$\mathbf{r}_0 = -\mathbf{D}^{-1}\,\mathbf{p}$$

where the components of the decentration are interpreted as the components of $\mathbf{r}_0$ while the prescribed prismatic power is given by **p**. The inverse of the dioptric power matrix is given by

$$\mathbf{D}^{-1} = \frac{1}{D_uD_v}\begin{pmatrix} D_{yy} & -D_{xy} \\ -D_{yx} & D_{xx} \end{pmatrix} \qquad (22)$$

HHF did not go into details of singular cases, that is $D_u = 0$ or $D_v = 0$. Furthermore, no attempt was made to include the effect that decentrations are referenced to the plane containing the centers of eye rotation, while $\mathbf{r}_0$ is often measured in the plane of the lens. For bifocal lenses HHF derived all the equations which deal with prismatic effects, mainly image jump at the dividing line and the segment center position. We do not consider details here, because these are specialized results derived from eq. (5). However, in his time these questions were among the most demanding in the lens spectacle business and HHF chose this topic to demonstrate the usefulness of the matrix approach.

We now proceed to some more far-reaching results on astigmatic systems. At the moment when the elements are no more in direct contact, but are separated by finite distances, new and unknown features show up. HHF restricted himself to systems with two elements. But his results are easily generalized to more complicated systems. These investigations are presented in the last part of PCMO and conclude the series. HHF introduced an imaging equation for one thin lens based on the application of Prentice's rule to each principal meridian. For an arbitrary ray Prentice's rule is applied and the direction changes are linked to the object and image distances *a* and *b* leading to an astigmatic imaging equation in matrix form

$$\mathbf{B} = \mathbf{A} + \mathbf{D} \qquad (23)$$

To this end HHF defined (Sect. 9.1) vergence matrices **A** and **B** in object and image space. The imaging equation expresses the fact that the change in vergence matrices, **B** – **A**, is determined by the dioptric power matrix.

Since HHF did not consider a single surface but a single *thin* astigmatic lens all refractive indices equal 1 and no reduced distances appear. However, the vergence matrices are more than the reciprocal values of (reduced) distances. When applied to the ray coordinates at the lens, $\mathbf{A}\mathbf{r}_0$, they determine the change of ray directions in an astigmatic bundle by

$$\frac{(\mathbf{r} - \mathbf{r}_0)}{z} = -\mathbf{A}\,\mathbf{r}_0 \qquad (24)$$

where z is the distance from the lens and **r** the vector of ray coordinates in that plane. For $z = 1$ m we get Prentice's rule. Equivalently, we can write

$$\mathbf{r}(z) = (1 - z\mathrm{A})\,\mathbf{r}_0 \qquad (25)$$

which corresponds to a kind of transfer equation. The identity matrix is denoted as **I**.

Having these tools at hand the generalized back-vertex power (better: back-vertex vergence) is now determined through a step by step method (9,31 ff). At the first lens and for a parallel beam (**A** = **0**) we have

$$\mathbf{r}_1'(z) = (\mathbf{I} - z_1\mathbf{D}_1)\,\mathbf{r}_{0,1} \qquad (26)$$

Chose $z_1 = \delta$, where δ denotes the distance between the two lenses, and we arrive at the ray vector at surface 2, given by

$$\mathbf{r}_{0,2} = (1 - \delta\mathbf{D}_1)\,\mathrm{r}_{0,1} \qquad (27)$$

The same ray bundle may be described by the perspective of lens 2, which leads to

$$\mathbf{r}_2 = (1 - z_2\mathbf{A}_2)\,\mathbf{r}_{0,2} \qquad (28)$$

Therefore, the relation

$$\mathbf{A}_2\,\mathbf{r}_{0,2} = \mathbf{D}_1\,\mathbf{r}_{0,1} \qquad (29)$$

holds. Replacing $\mathbf{r}_{0,2}$ by eq. (27) yields

$$\mathbf{A}_2\,(1 - \delta\mathbf{D}_1)\,\mathbf{r}_{0,1} = \mathrm{D}_1\,\mathbf{r}_{0,1} \qquad (30)$$

or after solving for $\mathbf{A}_2$

$$\mathbf{A}_2 = \frac{\mathbf{D}_1}{1 - \delta\mathbf{D}_1} \qquad (31)$$

The representation as a fraction is unambiguous since only symmetric matrices with parallel principal meridians are involved. This equation explicitly represents the transfer process for vergence matrices across a distance δ. Now the imaging equation is applied at lens 2

$$\mathbf{B}_2 = \mathbf{D}_2 + \mathbf{A}_2 \qquad (32)$$

Finally, the back-vertex power $\mathbf{S}'$ is given by

$$\mathbf{S}' = \mathbf{B}_2 = \mathbf{D}_2 + \frac{\mathbf{D}_1}{1 - \delta\mathbf{D}_1} \qquad (33)$$

The back-vertex power matrix $\mathbf{S}'$ results from an addition of symmetric matrices and thus is itself symmetric with two orthogonal principal directions. This matrix can always be decoded to the traditional SCA-values sphere, cylinder, and axis. Furthermore, the equation is a generalization of the well known equation for spherical lenses.

The difference between the equivalent power and the back-vertex power is explained by the concept of principal planes or principal points. Traditionally, the distance from the principal plane to the focal point is defined as focal length and the reciprocal value is known as equivalent power. However, for a thick astigmatic lens there exist no principal planes, since the incoming and outgoing rays need not intersect each other. Only two pairs of principal lines (object and image space) may exist. In general there is no ray, beside the optical axis, which passes through an arbitrary astigmatic lens system in one fixed plane. Therefore, we have no focal lengths nor an effective power.

To overcome this problem HHF defined the dioptric power matrix in section 9,4 as follows:

The dioptric power matrix D of an arbitrary system is a 2 × 2 matrix, which maps the ray vectors of all incoming parallel paraxial rays into the direction vectors of all outgoing rays. (34)

The dioptric power matrix is no longer defined by the reciprocal values of a focal length, but as a magnitude describing the direction changes of rays. Today this feature is realized by the lower-left matrix in the general 4 × 4 matrix representation. For a system of two elementswe already know the direction vector which is given by

$$\frac{\mathbf{r}_{02} - \mathbf{r}'_2}{z} = - \mathbf{S}'\mathbf{r}_{02}$$

The back-vertex dioptric power matrix connects ray coordinates of the *outgoing* rays with the related direction vectors. According to the above definition we seek a matrix which maps all incoming ray vectors, that is ray coordinates at the first surface, to the direction vectors of the outgoing rays

$$\mathbf{D}\,\mathbf{r}_{01} = \mathbf{S}'\mathbf{r}_{02}$$

With eq. (27) we have the required relation at hand yielding

$$\mathbf{D}\,\mathbf{r}_{01} = \mathrm{S}'\,(1 - \delta\mathbf{D}1)\,\mathbf{r}_{01}$$

After multiplication we arrive at the final result (9,45)

$$\mathbf{D} = \mathbf{D}_1 + \mathbf{D}_2 - \delta\mathbf{D}_2\mathbf{D}_1 \qquad (35)$$

or

$$\mathbf{D} = \mathbf{S}'\,(1- \delta\mathbf{D}_1)$$
$$= \left(\frac{\mathbf{D}_1}{(1 - \delta\mathbf{D}_1)} + \mathbf{D}_2 \right) (1 - \delta\mathbf{D}_1)$$

HHF commented his result in various ways: "The fact that the complicated ray tracing in a general astigmatic system of two elements can be described by an equation whose form is equivalent to the Gullstrand equation justifies the introduction of basic matrix algebra."

Furthermore, HHF recognized that the matrix **D** is not always symmetric. In case of an asymmetric matrix the difference of the principal values must not be interpreted as a cylinder because the principal meridians are not orthogonal. The analysis of the dioptric power matrix reveals that beside the known astigmatic refraction we have an additional torsion of the ray bundle. If the ray direction is reversed than a different dioptric power matrix $\tilde{\mathbf{D}}$ has to be applied, which is given by (9,57)

$$\tilde{\mathbf{D}} = \mathbf{D}_1 + \mathbf{D}_2 - \delta\mathbf{D}_1\mathbf{D}_2 \qquad (36)$$

Note that the order of the matrices in the last term has been reversed. The relation between the back-vertex power and the equivalent power in spherical systems is given by the shape factor $\mathbf{V}_E$, which describes the change of magnification due to the form of the lens. The shape factor can be redefined by the matrix

$$\mathbf{V}_E = \frac{1}{1 - \delta\mathbf{D}_1} \qquad (37)$$

yielding the relation (9,56) between the (asymmetric) power matrix of the whole system and the symmetric back-vertex power matrix

$$\tilde{\mathbf{D}}\,\mathbf{V}_E = \mathbf{S}' \qquad (38)$$

Without further derivations HHF gives results (9,51 ff) for so-called principal point matrices

$$\mathbf{H}' = \mathbf{s}' - \mathbf{f}' = \mathbf{S}'^{-1} - \mathrm{D}^{-1} = - \delta\mathbf{D}_1\mathbf{D}^{-1} \qquad (39)$$
$$\mathbf{H} = -\delta\mathbf{D}_2\tilde{\mathbf{D}}^{-1} \qquad (40)$$

and nodal point matrices

$$\mathbf{N}' = - \delta\tilde{\mathbf{D}}^{-1}\mathbf{D}_1 \qquad \mathbf{N} = \delta\mathbf{D}^{-1}\mathbf{D}_2$$

From these magnitudes HHF concludes that the imaging equation (23) must not be applied to the total system of lenses with finite distances. The imaging equation is restricted to a single element only.

Conclusion

HHF did an enormous job and arrived at the main results in the field of linear optics of astigmatic systems. He did it on his own, and he did everything from scratch. He lacked a community to discuss with and share and develop further ideas. He was, as it were, Robinson Crusoe on a dioptric power matrix island. He was not trained for the problems he had to solve.

Instead, he learned everything he needed along the way. And it seems that his power increased with the problems he encountered. In the end HHF knew that it would take some time before his ideas would be accepted and in the last chapter of PCMO he wrote in 1974: *The author was and is still aware that the use of matrices is obscure to most optometrists of the present generation. And only a few readers could follow the explanations to the very end. This will be very different in the coming generations. Already at some high schools the matrix notion and matrix algebra are introduced. Not long from now, everyone who graduates from college or even high school will be familiar with vector and matrix algebra. Those trained generations of optometrists will deal with vergence and power matrices more easily. And they will realize and appreciate the progress in the description of paraxial optics achieved by those concepts*.

This optimistic view might be proven true, since the matrix approach to linear optics is nearly a success story. HHF, who contributed so much to this discipline and aimed so much at the practical applications, passed away before his contributions were recognized. Nevertheless, HHF invested his work in a project whose time has now come.

Short biography of Hans-Heinrich Fick

Hans-Heinrich Fick was born on July 9th, 1922, in Stade near Hamburg in Germany. At the age of 15, in March 1938, HHF finished high school (Mittelschule). In September of the same year he started his first apprenticeship as an optician at the company Trüte in Bremen. After his examination in September 1941, he was drafted by the army and served until the end of World War II. He was a prisoner of war until the English army released him in summer 1945. The next three years he spent with an additional apprenticeship as a watchmaker at the company König in Bremervorde.

After his examination HHF continued working in his original profession as an optician. After some practical experience as a journeyman he applied in 1953 for a one year class (Meisterprüfung). HHF successfully finished the class at the Cologne School of Ophthalmic Opticians (Höhere Fachschule f¨ur Augenoptik, Köln) in March 1954. For one and a half years HHF worked as a "Meister" in Minden.

The last chapter of his career began in October 1955 when he followed a call to teach at the school where he himself graduated a year before. In June of 1957 he married Dorothea Fick who accompanied him all his life. The 32 year period at the Cologne School of Ophthalmic Opticians ended in 1986, when he officially retired. Nevertheless, he continued to publish and teach various classes organized by the Chamber of Handicrafts in Köln and preparation-courses by the Central Association of Ophthalmic Opticians (ZVA, Zentralverband der Augenoptiker) until he passed away at the age of 74 on April 19th in 1996.

HHF received several awards for his continued excellent work in the education of optometrists: in 1975 he received the Geerd-Marcus-award, which was followed by the WVAO Golden Needle of Honour in 1980. The ZVA presented him with the Silver Sign of Honour in 1985. After HHF's retirement the Leon-Hauck award for his lifelong work in the education of opticians was given to HHF in 1987.

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References

1. Hans-Heinrich Fick, Fortschrittliche Rech-nungsarten in der Augenoptik, Folge 1 bis 22, in: Der Augenoptiker **1**, 1972 - **6**, 1974.
2. My colleagues B Lingelbach and H. Diepes drew my attention to HHF in 1999.
3. Until 1982 in Germany an academic education in optometry was lacking. After the introduction of optometry into the Universities of Applied Sciences in Aalen and Jena, we have had, side by side, two types of optometrists. One is the academically educated engineer (Dipl.-Ing.). On the other hand we have the traditional optometrist with the highest grade called "Meister". HHF belonged obviously to the pre-academic area.